Calc 7.7: Improper integrals (1/16)

Improper limits

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx, \quad \text{if limit exists}$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \quad \text{if both integrals on right exist}$$

Improper integrand: if $f(x)$ is continuous on $a \le x < b$, but blows up as $x \to b$,

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx, \quad \text{if limit exists}$$

If $f(x)$ blows up at interior point c, $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad \text{if both integrals on right exist}$$

Calc 7.8: Comparison of Improper $\int$'s (1/18)

Comparison Test:

Let $f(x)$ and $g(x)$ be non-negative functions with $f(x) \leq g(x)$ for $x \geq c$.

If $\int_c^\infty g(x)\,dx$ converges, then $\int_c^\infty f(x)\,dx$ converges.

If $\int_c^\infty f(x)\,dx$ diverges, then $\int_c^\infty g(x)\,dx$ diverges.

There is a similar statement for improper integrals where the integrand blows up.

Useful comparison integrals:

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leq 1$.

$\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$ and diverges if $p \geq 1$.

$\int_0^\infty e^{ax}\,dx$ converges if $a < 0$ and diverges if $a \geq 0$.

Calc 9.1: Sequences (1/23)

A sequence is an infinite list of numbers, $s_1, s_2, s_3, \cdots$.
More precisely, it is a function from the natural numbers $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ to the real numbers $\mathbb{R}$.

Convergence: We say that a sequence s_n conveges to a real number L, if for every positive real number ϵ we can find a positive integer N such that $|s_n - L| < \epsilon$ whenever $n \geq N$. We say the sequence s_n diverges if there is no such L.

We say a sequence s_n is bounded if there are numbers K and M such that $K \leq s_n \leq M$ for $n = 1, 2, 3, \cdots$.

We say a sequence is increasing if $s_n \leq s_{n+1}$ for $n = 1, 2, 3, \cdots$.
It is decreasing if $s_n \geq s_{n+1}$ for $n = 1, 2, 3, \cdots$.
It is monotone if it is increasing or decreasing.

Theorem: Every convergent sequence is bounded.

Theorem: Every bounded monotone sequence converges.

Calc 9.2: Geometric Series (1/25)

A geometric series is a series of the form

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

If $|x| < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \neq 1$, then

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x}$$

Calc 9.3: Convergence of Series (1/28)

The partial sums of $\sum_{k=1}^{\infty} a_k$ are $s_n = \sum_{k=1}^{n} a_k$. The series converges to L if the sequence of partial sums converges to L.

Integral test: Suppose $f(x)$ is a decreasing, positive function for $x \geq c$ and $a_n = f(n)$.
If $\int_c^{\infty} f(x)\,dx$ converges, then $\sum_n a_n$ converges.
If $\int_c^{\infty} f(x)\,dx$ diverges, then $\sum_n a_n$ diverges.

Application: $\sum_n n^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Some properties:
1. If $\sum_n a_n$ and $\sum_n b_n$ both converge, then $\sum_n (ca_n + db_n)$ converges to $c\sum_n a_n + d\sum_n b_n$.
2. Changing a finite number of terms does not change whether a series converges.

Comparison test: Suppose $0 \leq a_n \leq b_n$.
If $\sum_n b_n$ converges, then $\sum_n a_n$ converges
If $\sum_n a_n$ diverges, then $\sum_n b_n$ diverges

Absolute convergence : We say $\sum_n a_n$ converges absolutely if $\sum_n |a_n|$ converges. If $\sum_n a_n$ converges absolutely then it converges.

Application: $\sum_n n^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Ratio test:
For the series $\sum_n a_n$, suppose the sequence $|a_{n+1}|/|a_n|$ conveges:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

If $L < 1$ then the series $\sum_n a_n$ converges.
If $L > 1$ then the series $\sum_n a_n$ diverges.

Calc 9.4: continued (2/1)

Limit comparison test:

Suppose $b_n > 0$, $a_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is nonzero.
Then $\sum_n a_n$ converges if and only if $\sum_n b_n$ converges.

Alternating series test:

Suppose $a_n > 0$, $a_{n+1} \le a_n$ and the sequence a_n converges to zero.
Then $\sum_n (-1)^n a_n$ converges.

Calc 9.5: Power series (2/4)

A power series about a is a series of the form

$$\sum_{n=1}^{\infty} c_n (x-a)^n$$

Radius of convergence Every power series has a radius of convergence $R \in [0, \infty]$ with the properties that
if $|x-a| < R$ then the series converges absolutely and
if $|x-a| > R$ then the series diverges.

Finding R: Use the ratio test. The condition $L < 1$ will give you a condition on x which will tell you R.

Complex power series: $\sum_{n=0}^{\infty} c_n z^n$. All of the above still holds with $|x+iy|$ defined to be $\sqrt{x^2+y^2}$.

Calc 10.1: Taylor polynomials (2/6)

The Taylor polynomial of degree n of $f(x)$ about a is

$$
\begin{aligned}
p_n(x) \quad & f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\
+ \quad & \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\
= \quad & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k
\end{aligned}
$$

where $f^{(k)}$ denotes the kth derivation of f.

It is the polynomial that best approximates f near $x = a$ in the sense that the value of this polynomial and its first n derivatives at $x = a$ agree with those of $f(x)$.

Calc 10.2: Taylor series (2/8)

The Taylor series of $f(x)$ about a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\
(1+x)^p &= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots
\end{aligned}$$

Calc 10.3: Shortcuts, applications of T.S. (2/11)

Products: Taylor series of fg is product of T.S. of f and T.S. of g.
For Taylor polynomial of degree n, use Taylor polynomials of degree n for f and g, BUT throw out terms of degree $> n$.

Substitution: For Taylor series of $f(x^p)$, substitute x^p into T.S. of $f(x)$.
Taylor polynomial of f of degree n will give Taylor polynomial of degree np for $f(x^p)$ when you do this.

Composition: If $g(0) = 0$ we can get T.S. about 0 of $f(g(x))$ by substituting T.S. of $g(x)$ for the argument in T.S. of $f(x)$.
For polynomials, again throw out terms you get with degree $> n$.

Integration/differentiation: You can integrate or differentiate Taylor series you already know to generate new ones.

Calc 10.4: Error and convergence (2/15)

Error for nth order Taylor polynomial:

$$|f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}$$

M_{n+1} is max of $|f^{(n+1)}|$ over interval from a to x.

Series convergence: For a given x, if you can show the error in the Taylor polynomial goes to 0 as $n \to \infty$, then at x, the Taylor series converges to the function.

Dif Eq 3.5: Power series solutions (2/18)

Power series solution of dif eq about 0: We assume the solution has a power series expansion about 0:

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} c_n x^n \\
y'(x) &= \sum_{n=1}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n
\end{aligned}
$$

Find the Taylor expansions of any functions of x in the equation.

Replace everybody in the dif. eq. by their power series.

Each order (x^n) gives an equation for the c's.

Usually, nth order gives an equation for c_n in terms of c_k, $k < n$.

Without an initial condition, there is a free parameter, often c_0.

Dif Eq 6.1: Phase plane analysis (2/25)

Autonomous system of first order equations:

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y) \end{aligned}$$

At point (x,y), graph the vector $(P(x,y),Q(x,y))$.
Solutions must follow this vector field

Equilibrium solutions:
$x(t) = x_0$, $y(t) = y_0$ is a solution if
$P(x_0,y_0) = 0, Q(x_0,y_0) = 0$.

A system of first order differential equations:

$\frac{dx}{dt} = P(t, x, y), \quad \frac{dy}{dt} = Q(t, x, y)$

Initial condition is $x(t_0) = x_0$, $y(t_0) = y_0$.

t_0, x_0, y_0 are constants.

Second order differential equation:

$\frac{d^2x}{dt^2} = R(t, x, \frac{dx}{dt})$

Initial conditions: $x(t_0) = x_0$, $\frac{dx}{dt}(t_0) = x_0^*$

Boundary value problem: $x(a) = x_0$, $x(b) = x_1$

Second order dif eq as first order system:

Let $y = \frac{dx}{dt}$ and the second order dif. eq. can be written as the first order system

$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = R(t, x, y)$

This means that techniques for first order systems can be applied to second order dif. eqs.

Dif Eq 6.3: Const coef, hom., linear 2nd order

Second order homogeneous linear dif. eq. with constant coefs:
$ax'' + bx' + cx = 0$
Solve the characteristic equation: $ar^2 + br + c = 0$.

Two distinct real roots ($b^2 - 4ac > 0$):
Let r_1, r_2 be the roots. General solution is
$c_1 \exp(r_1 t) + c_2 \exp(r_2 t)$

Two complex roots ($b^2 - 4ac < 0$):

Let $\alpha \pm i\beta$ be the roots. General solution is
$c_1 \exp(\alpha t) \sin(\beta t) + c_2 \exp(\alpha t) \cos(\beta t)$

One double real root ($b^2 - 4ac = 0$):

Let r be the root. General solution is
$c_1 \exp(rt) + c_2 t \exp(rt)$

Dif Eq : More on autonomous systems (3/5)

Consider an autonomous first order system

$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y)$

Equilibrium solutions:

If (x_0, y_0) is such that $P(x_0, y_0) = 0, Q(x_0, y_0) = 0,$
then $x(t) = x_0$, $y(t) = y_0$ is an equilibrium solution.

Trajectories in phase plane :

Try to find an equation for the curve $(x(t), y(t))$ in the $x - y$ plane.

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{y'}{x'} = \frac{Q(x, y)}{P(x, y)}$$

Solve this dif. eq.

NB: this only gives the trajectory, not the time dependence of the solution.

7.1: General solutions - linear dif. eq. (3/12)

Linear homogeneous: $a_2(t)x'' + a_1(t)x' + a_0(t)x = 0$
General solution: $x(t) = c_1x_1(t) + c_2x_2(t)$
where $x_1(t)$ and $x_2(t)$ are different solutions of dif. eq.

Linear (inhomogeneous) : $a_2(t)x'' + a_1(t)x' + a_0(t)x = f(t)$
General solution: $x(t) = x_p(t) + c_1x_1(t) + c_2x_2(t)$
where $x_p(t)$ is a particular solution of the inhomogeneous dif. eq. and $x_1(t)$ and $x_2(t)$ are different solutions of homogeneous dif. eq.

7.2: Undetermined coefficients (3/12)

Constant coef, linear:

$$a_2 x'' + a_1 x' + a_0 x = f(t)$$

where a_0, a_1, a_2 are constants. This is a method for finding a particular solution by guessing.

$$f(t) = ae^{ct}, \quad guess = Ae^{ct}$$

$$f(t) = b_0 + b_1 t + \cdots + b_n t^n, \quad guess = B_0 + B_1 t + \cdots + B_n t^n$$

$$f(t) = a\sin(\omega t) + b\cos(\omega t), \quad guess = A\sin(\omega t) + B\cos(\omega t)$$

$$f(t) = (b_0 + b_1 t + \cdots + b_n t^n)e^{ct}, \quad guess = (B_0 + B_1 t + \cdots + B_n t^n)e^{ct}$$

$$f(t) = (b_0 + \cdots + b_n t^n)\sin(\omega t)e^{ct} + (c_0 + \cdots + c_n t^n)\cos(\omega t)e^{ct},$$
$$guess = (B_0 + \cdots + B_n t^n)\sin(\omega t)e^{ct} + (C_0 + \cdots + C_n t^n)\cos(\omega t)e^{ct}$$

If guess gives 0 in place of $f(t)$, try multiplying guess by t.

9.1: Constant coef linear systems (3/28)

Constant coef, linear system:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

Equilibrium at $(0, 0)$.

To solve it, compute x'', eliminate y', y to get second order dif eq for x. Solve it by characteristic equation.

Stable node (sink) : Real roots, both < 0.

Unstable node (source) : Real roots, both > 0.

Saddle : Real roots, one < 0, one > 0.

Stable spiral (focus) : Complex roots $\alpha \pm i\beta$ with $\alpha < 0$.

Unstable spiral (focus) : Complex roots $\alpha \pm i\beta$ with $\alpha > 0$.

Periodic (center) : Complex roots $\alpha \pm i\beta$ with $\alpha = 0$.

9.5: Linear algebra approach (4/7)

Multiplication; determinant

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}, \quad det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are roots of $det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$.

Eigenvector for λ is a solution $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Given eigenvalue λ and eigenvector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a solution of

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

8.2: Reduction of order (4/14)

Want to solve

$$a_2(t)x''(t) + a_1(t)x'(t) + a_0(t)x(t) = f(t)$$

given a solution $x_h(t)$ of

$$a_2(t)x_h''(t) + a_1(t)x_h'(t) + a_0(t)x_h(t) = 0$$

The trick: Look for a solution of the form $x(t) = x_h(t)z(t)$.

The miracle: The dif. eq. for z will involve z' and z'', but not z.
So it is really a first order equation for z'.
Solve for z', then for z and finally for x.

Special case: $f(t)$ can be 0. So you can use one solution of the homogeneous equation to find another solution of the homogeneous equation.

8.3: Variation of parameters (4/16)

Want to solve

$$a_2(t)x''(t) + a_1(t)x'(t) + a_0(t)x(t) = f(t)$$

given two solutions $x_1(t)$ and $x_2(t)$ of the homogeneous eq.

$$a_2(t)x_h''(t) + a_1(t)x_h'(t) + a_0(t)x_h(t) = 0$$

The trick: Look for a solution of the form $x(t) = x_1(t)z_1(t) + x_2(t)z_2(t)$.
This will work provided $z_1(t)$ and $z_2(t)$ satisfy

$$\begin{aligned} z_1'x_1 + z_2'x_2 &= 0 \\ z_1'x_1' + z_2'x_2' &= \frac{f}{a_2} \end{aligned}$$

Solve the above two equations for the two unknowns z_1', z_2'. Then integrate to get z_1 and z_2.

10.1: Non-linear autonomous systems (4/21)

We can draw phase planes for

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y) \end{aligned}$$

Equilibrium solutions: $x(t) = x_0$, $y(t) = y_0$ where constants x_0, y_0 simultaneously satisfy $P(x_0, y_0) = 0, \quad Q(x_0, y_0) = 0$.

Periodic solutions correspond to closed trajectories or orbits in the phase plane.

For a stable equilibrium the basin of attraction is the set of points that flow to the equilibrium.

Trajectories or orbits solve $\frac{dy}{dx} = \frac{y'}{x'}$

A separatrix is a trajectory or orbit that separates qualitatively different behaviors of the solutions. Typically a separatrix will start or end at an equilibrium or go between two equilibria.

10.3: Linearization (4/23)

Let (x_0, y_0) be an equilibrium point.

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Let (x_0, y_0) be an equilibrium point.

The linearized system: about (x_0, y_0) is

$$
\begin{aligned}
x' &= P_x(x_0,y_0)(x-x_0) + P_y(x_0,y_0)(y-y_0) \\
y' &= Q_x(x_0,y_0)(x-x_0) + Q_y(x_0,y_0)(y-y_0)
\end{aligned}
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Subscripts x, y mean partial derivatives with repect to x, y.

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Linearization theorem: The nature (saddle, stable/unstable node, stable/unstable focus, center) of the non-linear system is the same as that of the linear system with the following exceptions.

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