Math 250b (Spring '08) - Homework 13 Part I -Solutions

1. For each of the following three autonomous non-linear systems,

(i) Find all the equilibrium points.

(ii) Find the linear systems that approximate the original system near each equilibrium.

(iii) State what you can conclude from the linearization theorem for each equilibrium in the original system.

(iv) Use P-Plane to plot a bunch of trajectories and then print it and indicate on the plot where the separatrices are.

(a)

$$\begin{aligned} x' &= y \\ y' &= x(x^2-1) \end{aligned}$$

Equilibria are at $(0,0),(1,0),(-1,0)$. Jacobian is

$$J(x,y) = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3x^2-1 & 0 \end{pmatrix}$$

At $(0,0)$ this becomes

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

The roots are given by

$$0 = det \begin{pmatrix} -r & 1 \\ -1 & -r \end{pmatrix} = r^2+1$$

So they are $\pm i$. The linearization theorem only tells us that for the original system $(0,0)$ can be a center, stable spiral or unstable spiral.

At $(1,0)$

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= y \\ y' &= 2(x-1) \end{aligned}$$

The roots are given by

$$0 = det \begin{pmatrix} -r & 1 \\ 2 & -r \end{pmatrix} = r^2 - 2$$

So the roots are $\pm\sqrt{2}$. The linearization theorem says the original system must have a saddle at $(1,0)$.

At $(-1,0)$

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= y \\ y' &= 2(x+1) \end{aligned}$$

The roots are given by

$$0 = det \begin{pmatrix} -r & 1 \\ 2 & -r \end{pmatrix} = r^2 - 2$$

So the roots are again $\pm\sqrt{2}$. The linearization theorem says the original system must have a saddle at $(-1,0)$.

(b)

$$\begin{aligned} x' &= y^2 - x^2 \\ y' &= 2 - e^x \end{aligned}$$

Equilibria are at $(\ln 2, \ln 2), (\ln 2, -\ln 2)$. Jacobian is

$$J(x,y) = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} -2x & 2y \\ -e^x & 0 \end{pmatrix}$$

At $(\ln 2, \ln 2)$,

$$J(\ln 2, \ln 2) = \begin{pmatrix} -2\ln 2 & 2\ln 2 \\ -2 & 0 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= -2\ln 2(x - \ln 2) + 2\ln 2(y - \ln 2) \\ y' &= -2(x - \ln 2) \end{aligned}$$

Computing the roots is a mess, but you find they are complex with a negative real part. So the linearization theorem says it is a stable spiral.

At $(\ln 2, -\ln 2)$.

$$J(\ln 2, -\ln 2) = \begin{pmatrix} -2\ln 2 & -2\ln 2 \\ -2 & 0 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= -2\ln 2(x - \ln 2) - 2\ln 2(y + \ln 2) \\ y' &= -2(x - \ln 2) \end{aligned}$$

Computing the roots is a mess, but you find they are real with one positive and one negative. So the linearization theorem says it is a saddle.

(c)

$$\begin{aligned} x' &= e^{xy} - 1 \\ y' &= x + y^2 - 1 \end{aligned}$$

For an equilibrium, $e^{xy} = 1$ and $x + y^2 - 1 = 0$. The first equation implies $xy = 0$. So $x = 0$ or $y = 0$. If $x = 0$ then the second equation implies $y = \pm 1$. If $y = 0$ then the second equation implies $x = 1$. So we have three equilibria: $(0,1), (0,-1), (1,0)$. Jacobian is

$$J(x,y) = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} ye^{xy} & xe^{xy} \\ 1 & 2y \end{pmatrix}$$

At $(0,1)$,

$$J(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= x \\ y' &= x + 2(y-1) \end{aligned}$$

The roots are $1, 2$, so it is an unstable node.

At $(0,-1)$,

$$J(0,-1) = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= x \\ y' &= x - 2(y+1) \end{aligned}$$

The roots are $-1, -2$, so it is a stable node.

At $(1,0)$,

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So the linear system is

$$\begin{aligned} x' &= y \\ y' &= x - 1 \end{aligned}$$

The roots are $-1, 1$, so it is a saddle.