Sample Exam 2 Answers - Math 464/564 - Spring 07 -Kennedy

1. (a) Using law of the unconscious mathematician,

$$\mu = E[Y] = \int_{-1}^{1} x^3 \, \frac{1}{2} dx = 0$$

Since the mean is zero, the variance is

$$\sigma^2 = E[Y^2] = \int_{-1}^{1} x^6 \, \frac{1}{2} \, dx = \frac{1}{7}$$

(b) Note that the range of Y is $[-1, 1]$. Compute the cdf and then differentiate it.

$$f_Y(y) = \frac{1}{6} y^{-2/3}, \qquad -1 \leq y \leq 1$$

2. (a) First compute the marginals:

$$f_X(x) = \int_0^1 \frac{3}{2}(x^2 + y^2) \, dy = \frac{1}{2} + \frac{3}{2}x^2, \qquad 0 \leq x \leq 1$$

$$f_Y(y) = \frac{1}{2} + \frac{3}{2}y^2, \qquad 0 \leq y \leq 1$$

Their product is not $f_{X,Y}(x, y)$, so X and Y are not independent.
(b)

$$P(X \leq Y) = \int_0^1 \left(\int_0^y \frac{3}{2}(x^2 + y^2) \, dx \right) \, dy = \frac{1}{2}$$

$$P(2X \leq Y) = \int_0^1 \left(\int_0^{y/2} \frac{3}{2}(x^2 + y^2) \, dx \right) \, dy = \frac{13}{64}$$

3. To test independence, we need the marginals:

$$
\begin{aligned}
f_X(j) &= \sum_{k=0}^{m} \binom{n}{j}\binom{m}{k} p^j (1-p)^{n-j} q^k (1-q)^{m-k} \\
&= \binom{n}{j} p^j (1-p)^{n-j} \sum_{k=0}^{m} \binom{m}{k} q^k (1-q)^{m-k} \\
&= \binom{n}{j} p^j (1-p)^{n-j}
\end{aligned}
$$

Thus X is binomial with n trials and probability p of success. Similarly,

$$
f_Y(k) = \binom{m}{k} q^k (1-q)^{m-k}
$$

Thus Y is binomial with m trials and probability q of success. The product of the marginal pmf's is the joint pmf, so they are independent.
(b)

$$
E[Z] = E[X] + E[Y] = np + mq
$$

Using the independence of X and Y,

$$
var(Z) = var(X) + var(Y) = np(1-p) + mq(1-q)
$$

4. (a)

$$
M_X(t) = \exp(t + \lambda(e^t - 1))
$$

(b) $\mu = \lambda + 1$, $\sigma^2 = \lambda$