Sample Final Exam Answers- Math 464/564 - Spring 07 -Kennedy

1. (a) 5/24 (b) 3/5

2. (a) 7/8 (b) 11/7.

3. (a)

	Y=0	Y=1	Y=2
X=0	16/36	8/36	1/36
X=1	8/36	2/36	0
X=2	1/36	0	0

(b) No. For example, $P(X=2, Y=2) = 0$. But $P(X=2) > 0$ and $P(Y=2) > 0$, so $P(X=2, Y=2) \neq P(X=2)P(Y=2)$.
(c) $E(XY) = 1/18$.

4.

$$f_Y(y) = \frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\, e^{-y^2/2}, \quad y \geq 0$$

5. (a)

$$f_X(x) = \frac{6}{7}x^2 + \frac{6}{7}x + \frac{2}{7}, \quad 0 \leq x \leq 1$$

$$f_Y(y) = \frac{6}{7}y^2 + \frac{6}{7}y + \frac{2}{7}, \quad 0 \leq y \leq 1$$

(b) No, X and Y are not independent. $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$.

6. (a) $P(Z < 1/2) = 1/4$
(b) The cdf is $F_Z(z) = z^2, \quad 0 \leq z \leq 1$. So the pdf is $f_Z(z) = 2z, \quad 0 \leq z \leq 1$, and $f_Z(z) = 0$ for z outside $[0,1]$.

7. (a) Differentiate the equation and then set $t = 0$ to find $M'(0) = 0$. So the mean is zero.
(b) Differentiate the equation and then set $t = 0$ to find $M'(0) = M(0)/2$. Since $M(0) = 1$ for any RV, the mean is 1/2.

8. (a) The Jacobian works out to 1/2. The region $x, y \geq 0$ is mapped to the region in the right half plane bounded by the lines $v = u$ and $v = -u$. This region may be written as $|v| \leq u$.

$$f_{U,V}(u, v) = \frac{1}{2} e^{-u}, \quad |v| \leq u$$

(b) You can use the joint pdf of U, V to compute it. Or just use $E[UV] = E[X^2 - Y^2] = E[X^2] - E[Y^2] = 0$.
(c) The marginals work out to

$$\begin{aligned} f_U(u) &= ue^{-u}, \quad u \geq 0 \\ f_V(v) &= \frac{1}{2} e^{-|v|}, \quad -\infty < v < \infty \end{aligned} \tag{1}$$

So their product is not equal to the joint pdf and so U and V are not independent.

9. Let X and Y be independent random variables, each of which has the standard normal density. Let $Z = 2X + Y - 5$.
(a) We know that the sum of independent normals is normal, and if you shift a normal RV it is still normal. So Z has a normal distribution with $\mu = -5$, $\sigma^2 = 4 + 1 = 5$. So

$$f_Z(z) = \frac{1}{\sqrt{10\pi}} \exp\left(-\frac{(x+5)^2}{10}\right)$$

(b) $E[Z|X] = E[2X + Y - 5|X] = 2X + E[Y] - 5 = 2X - 5$

10. (a) X_i has mean 1 and variance 2, so X has mean n and variance $2n$.
(b) Note : in an earlier version I mistakenly forgot to take the square root of the variance in the following. $0.95 = P(X - n < c) = P(\frac{X-n}{\sqrt{2n}} < \frac{c}{\sqrt{2n}}) \approx P(Z < \frac{c}{\sqrt{2n}})$ This implies $P(|Z| < \frac{c}{\sqrt{2n}}) = 0.9$, so $c/\sqrt{2n} = 1.64$ and so $c = 1.64\sqrt{2n}$. (Here Z is a standard normal.)