Sample Final Exam - Math 464 - Fall 2010 -Kennedy

1. A, B, C are events with $P(A) = 0.5$, $P(A \cup B) = 0.7$ and $P(C) = 0.2$. A and B are independent. A and C are disjoint. Find the probabilities
(a) $P(A^c|C)$. (As always, A^c denotes the complement of A.)
(b) $P(B)$

2. I flip a fair coin. If it is heads I roll a four-sided die, and if it is tails I roll a six-sided die.
(a) What is the probability the die roll is 3.
(b) If the die roll is 3, what is the probability the coin flip showed heads?
(c) Let X be the number that comes up on the die. Find the mean and variance of X.

3. I flip a fair coin until I get heads.
(a) Find the probability it takes at most 3 flips (including the flip that comes up heads).
(b) Let X be the number of flips it takes (including the flip that gives heads). Compute $E[X|X \leq 3]$, the expected value of X given that X is at most 3.

4. We roll two ordinary six-sided dice. Let X be the number of dice showing a 1 and Y the number of dice showing a 2. So each of X and Y can be $0, 1$ or 2.
(a) Find the joint density of X, Y.
(b) Are X and Y independent. Justify your answer.
(c) Compute $E(XY)$.

5. Let X have the gamma distribution with $\lambda = 1/2$ and $w = 1/2$. Find the p.d.f. of $Y = \sqrt{X}$.

6. Let X and Y be continuous random variables with joint density

$$f(x, y) = \begin{cases} \frac{6}{7}(x+y)^2, & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Find the marginal densities of X and Y.
(b) Are X and Y independent ?

7. Let X and Y be independent random variables. Each has the exponential distribution with $\lambda = 1$. Let $Z = Y - X$.
(a) Find the mean and variance of Z.
(b) Find the pdf of Z.

8. (a) Suppose X is a random variable whose moment generating function $M(t)$ satisfies the equation $M(t) = M(-t)$. Show that the mean of X is 0.
(b) Now suppose the moment generating function $M(t)$ satisfies the equation $M(t) = e^t M(-t)$. Find the mean of X.

9. Let X and Y be independent standard normal random variables. Define two new random variables by

$$\begin{aligned} U &= X^2 + Y^2 \\ W &= X^2 - Y^2 \end{aligned}$$

(a) Find the joint pdf of U, W.
(b) Are U and W independent random variables? Justify your answer.

10. Let X and Y be independent random variables, each of which has the standard normal density. Let $Z = 2X + Y - 5$.
(a) Find the pdf of Z.
(b) Find $E[Z|X = x]$.

11. Let $X_1, X_2, \cdots X_n$ be independent continuous random variables. They are identically distributed, i.e., they have the same distribution. Suppose that $EX_i = 1$ and $EX_i^2 = 3$. Define

$$X = \sum_{i=1}^{n} X_i \tag{2}$$

(a) Find the mean and variance of X.
(b) If Z has a standard normal distribution, then $P(|Z| < 1.96) = 0.95$ and $P(|Z| < 1.64) = 0.90$. Find c so that for large n, $P(X - n < c)$ is approximately 0.95. Your answer should depend on n. (Note that it is $X - n$ and not $|X - n|$.)

12. X and Y are discrete random variables with joint pmf

$$f_{X,Y}(x, y) = \frac{\lambda^{x+y} e^{-2\lambda}}{x!\, y!}$$

where x and y both take on the values $0, 1, 2, 3, \cdots$, and λ is a positive parameter.
(a) Find the marginal pmf's of X and Y.
(b) Find $P(X + Y = k)$. Simplify your answer; don't leave it as a sum.