Review questions for Exam 1 - Solutions Math 464 - Fall 18

1. Let X be a discrete RV with a Poisson distribution whose mean is 3.

(a) Find $\mathbf{P}(X \geq 1|X \leq 2)$.

Since the mean is 3, the parameter $\lambda = 3$.

$$\begin{aligned}\mathbf{P}(X \geq 1|X \leq 2) &= \frac{\mathbf{P}(1 \leq X \leq 2)}{\mathbf{P}(X \leq 2)} = \frac{\mathbf{P}(X=1)+\mathbf{P}(X=2)}{\mathbf{P}(X=0)+\mathbf{P}(X=1)+\mathbf{P}(X=2)} \\ &= \frac{e^{-3}(3+\frac{1}{2}3^2)}{e^{-3}(1+3+\frac{1}{2}3^2)} = \frac{15}{17}\end{aligned}$$

(b) Let $Z = 2X + 5$. Find the mean and variance of Z.

$\mathbf{E}[Z] = 2\mathbf{E}[X] + 5 = 2 \times 3 + 5 = 11$.
$var(Z) = 4var(X) + var(5) = 4 \times 3 + 0 = 12$.

2. A and B are events with $\mathbf{P}(A) = 0.1$, $\mathbf{P}(B|A) = 0.7$, $\mathbf{P}(B|A^c) = 0.2$.

(a) Find $\mathbf{P}(B)$. Since $\mathbf{P}(B|A) = \mathbf{P}(A \cap B)/\mathbf{P}(A)$ we have $\mathbf{P}(A \cap B) = 0.07$. And since $\mathbf{P}(B|A^c) = \mathbf{P}(A^c \cap B)/\mathbf{P}(A^c)$ and $\mathbf{P}(A^c) = 0.9$, we have $\mathbf{P}(A^c \cap B) = 0.18$. So $\mathbf{P}(B) = \mathbf{P}(A^c \cap B) + \mathbf{P}(A \cap B) = 0.25$.

(b) Find $\mathbf{P}(A|B)$. $\mathbf{P}(A|B) = \mathbf{P}(A \cap B)/\mathbf{P}(B) = 0.07/0.25 = 0.28$.

(c) Are A and B independent ? Justify your answer. Since $\mathbf{P}(A|B) \neq \mathbf{P}(A)$, they are not independent. Or you can check that $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$.

3. Each day a weatherman makes one of three predictions: "rain", "no rain", or "possibility of rain." The percentages of the time he makes each prediction are 10%,75%,and 15% respectively. If the weather forecast is for "rain," the probability it will rain is 70%. If the forecast is "no rain," the probability it will rain is 20%. If the forecast is for "possibility of rain", the probability of rain is 50%.

Let "FR","FP",and "FNR" denote the events that the weatherman predicted ... rain, possibility of rain or no rain, respectively. Let "R" "NR" denote the events that it actually rained or did not rain.

(a) Find the percentage of days on which it rains.

$$\begin{aligned}P(R) &= P(R|FR)P(FR) + P(R|FP)P(FP) + P(R|FNR)P(NR) \\ &= 0.7 * 0.1 + 0.5 * 0.15 + 0.2 * 0.75 = 0.295\end{aligned}$$

(b) Suppose it did not rain yesterday. What is the probability the forecast for yesterday was for "no rain?"

$$\begin{aligned} P(FNR|NR) &= \frac{P(FNR \cap NR)}{P(NR)} = \frac{P(NR|FNR)P(FNR)}{P(NR)} \\ &= \frac{(1-0.2)*0.75}{1-0.295} = \frac{0.6}{0.705} = 0.851 \end{aligned}$$

4. Let X and Y be independent random variables. They are identically distributed and have the geometric distribution with $p = 1/3$.

(a) What is the joint pmf of X and Y ?

Since they are independent

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) = \frac{1}{9}\left(\frac{2}{3}\right)^{x+y-2}$$

where $x, y = 1, 2, 3, \cdots$.

(b) Let $W = X + Y$. Find $\mathbf{P}(W = 3)$.

$$\mathbf{P}(W=3) = \mathbf{P}(X+Y=3) = \mathbf{P}(X=1, Y=2) + \mathbf{P}(X=2, Y=1) = \frac{2}{27} + \frac{2}{27} = \frac{4}{27}$$

(c) Let $Z = X - Y$. Find $\mathbf{P}(Z = 0)$.

Note that geometric RV's start at 1, not 0. So

$$\mathbf{P}(Z=0) = \mathbf{P}(X=Y) = \sum_{n=1}^{\infty} \mathbf{P}(X=n, Y=n) = \sum_{n=1}^{\infty} p_{X,Y}(n,n) = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{2}{3}\right)^{2n-2}$$

Note that $(2/3)^{2n-2} = (4/9)^{n-1}$. So this is

$$= \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{4}{9}\right)^{n-1} = \sum_{k=0}^{\infty} \frac{1}{9}\left(\frac{4}{9}\right)^{k} = \frac{1}{9}\frac{1}{1-\frac{4}{9}} = \frac{1}{5}$$

(d) $Z = X - Y$. Find the mean and variance of Z. Note that X and Y both have mean $1/p = 3$ and both have variance $(1-p)/p^2 = 6$. The mean of Z is $E[Z] = E[X] - E[Y] = 0$. Since X and Y are independent, $var(Z) = var(X) + var(Y) = 6 + 6 = 12$.

(e) Find $\mathbf{E}[XY^2]$.

Since they are independent $\mathbf{E}[XY^2] = \mathbf{E}[X]\mathbf{E}[Y^2]$. For geometric, $\mathbf{E}[X] = 1/p = 3$. And $var(Y) = (1-p)/p^2 = 6$. So $\mathbf{E}[Y^2] = var(Y) + \mathbf{E}[Y]^2 = 6 + 3^2 = 15$. So $\mathbf{E}[XY^2] = 45$

5. N is a discrete random variable with range $\{1, 2, 3, 4\}$. It has the uniform distribution. An unfair coin has probability p of heads. We flip the coin N times and let X be the number of heads we get.

(a) Find $\mathbf{P}(X = 3)$.

We use the partition theorem with partition $N = n$ where $n = 1, 2, 3, 4$.

$$\mathbf{P}(X = 3) = \sum_{n=1}^{4} \mathbf{P}(X = 3|N = n)\,\mathbf{P}(N = n)$$

Note that $\mathbf{P}(X = 3|N = n) = 0$ for $n = 1, 2$. For $n = 3, 4$, the distribution of X given $N = n$ is binomial:

$$\mathbf{P}(X = 3|N = 3) = \binom{3}{3} p^3 (1-p)^0 = p^3,$$
$$\mathbf{P}(X = 3|N = 4) = \binom{4}{3} p^3 (1-p)^1 = 4p^3(1-p)$$

Also, $\mathbf{P}(N = n) = 1/4$. So

$$\mathbf{P}(X = 3) = \left[p^3 + 4p^3(1-p)\right]/4 = \frac{p^3}{4} + p^3(1-p)$$

(b) Find the mean and variance of X.

For the mean we use the partition theorem again with the same partition.

$$\mathbf{E}[X] = \sum_{n=1}^{4} \mathbf{E}[X|N = n]\,\mathbf{P}(N = n)$$

Since X given $N = n$ is binomial, $\mathbf{E}[X|N = n] = np$. So

$$\mathbf{E}[X] = \sum_{n=1}^{4} np\,\mathbf{P}(N = n) = p\mathbf{E}[N] = p\frac{1+2+3+4}{4} = \frac{5}{2}p$$

For the variance the partition does NOT say that

$$var(X) = \sum_{n=1}^{4} var(X|N = n)\, \mathbf{P}(N = n)$$

and in fact this equality is not true. But we do have

$$\mathbf{E}[X^2] = \sum_{n=1}^{4} \mathbf{E}[X^2|N = n]\, \mathbf{P}(N = n)$$

Given $N = n$, X is binomial so its variance is $np(1-p)$, and so $\mathbf{E}[X^2|N = n] = np(1-p) + n^2p^2$. So

$$\begin{aligned}
\mathbf{E}[X^2] &= \sum_{n=1}^{4} [np(1-p) + n^2p^2]\, \mathbf{P}(N = n) = p(1-p)\mathbf{E}[N] + p^2\mathbf{E}[N^2] \\
&= p(1-p)\frac{1+2+3+4}{4} + p^2\frac{1^2+2^2+3^2+4^2}{4} \\
&= \frac{5}{2}p(1-p) + \frac{15}{2}p^2 = \frac{5}{2}p + 5p^2
\end{aligned}$$

So

$$var(X) = \frac{5}{2}p + 5p^2 - \left(\frac{5}{2}p\right)^2 = \frac{5}{2}p - \frac{5}{4}p^2$$

6. This question was a duplicate of the previous question by mistake.

7. Die A has twelve faces, 9 of which are green and 3 of which are white. Die B has twelve faces, 6 of which are green and 6 of which are white. I flip a fair coin. If it is heads, I roll die A, and for tails I roll die B.

Let H and T be the events that the coin flip is heads and tails, respectively. Let G be the event that the die shows green . Drawing a tree is a good way to do this problem.

(a) Find the probability the face on the top of the die is green.

$$P(G) = P(G|H)P(H) + P(G|T)P(T) = \frac{3}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{5}{8}. \qquad (1)$$

(b) If green turns up, what is the probability die A was rolled?

$$P(H|G) = \frac{P(H \cap G)}{P(G)} = \frac{P(G|H)P(H)}{P(G)} = \frac{3/8}{5/8} = \frac{3}{5} \tag{2}$$

8. I have 10 one-dollar bills, 6 five-dollar bills and 2 ten-dollar bills.

(a) If I randomly arrange the bills in a row, what is the probability that as I look at them from left to right I first see all the one-dollar bills, then all the five-dollar bills and finally the ten-dollar bills?

To have equally likely outcomes we think of the 18 bills as being distinguishable. The number of ways to line them all up with no constraints is 18!. For the event in question, there are 10! ways to line up the one's, 6! ways for the fives and 2! for the tens. So probability is $10!\,6!\,2!/18!$.

(b) If I put all the bills in a hat and draw three, what is the probability the total value is no greater than 7$? The order of the three chosen doesn't matter. So the sample space has $\binom{18}{3}$ outcomes in it. To get 7 or less, we must either get 3 one's or 2 one's and a five. There are $\binom{10}{3}$ ways to get the former, $\binom{10}{2}\binom{6}{1}$ for the latter. Note that these are two separate cases, so we add these two numbers. Probability is

$$\frac{\binom{10}{3} + \binom{10}{2}\binom{6}{1}}{\binom{18}{3}}$$

(c) In how many ways can I give all of the one-dollar bills to 3 friends if the bill are considered to be identical? This is like putting 10 identical balls (the one dollar bills) into 3 distinguishable bins (the friends). Number of ways is

$$\frac{(10+3-1)!}{10!(3-1)!} = \frac{12!}{10!2!}$$

9. Consider a simple trial with two outcomes (success and failure) which is repeated $2n$ times with the repetitions being independent of one another. Let X be the total number of successes out the $2n$ reps, and Y the number of successes out of the first n reps.

(a) Find the joint p.m.f.of X, Y. Let Z be the number of successes in the last n trials. Then $X = Y + Z$. Note that Y and Z are independent since the trials are independent. And both Y and Z are binomial with n trial and parameter p. So

$$\begin{aligned}\mathbf{P}(Y=y,Z=z) &= \binom{n}{y}p^y(1-p)^{n-y}\binom{n}{z}p^z(1-p)^{n-z} \\ &= \binom{n}{y}\binom{n}{z}p^{y+z}(1-p)^{2n-y-z}\end{aligned}$$

So

$$\begin{aligned}\mathbf{P}(X=x,Y=y) &= \mathbf{P}(Y=y,Z=x-y) \\ &= \binom{n}{y}\binom{n}{x-y}p^x(1-p)^{2n-x}\end{aligned}$$

(b) Compute $\mathbf{E}[XY]$.

$$\mathbf{E}[XY]=\mathbf{E}[(Y+Z)Y]=\mathbf{E}[Y^2]+\mathbf{E}[ZY]=\mathbf{E}[Y^2]+\mathbf{E}[Z]\mathbf{E}[Y]$$

Since X and Y are binonial with n,p, $\mathbf{E}[Z]=\mathbf{E}[Y]=np$. And $var(Y)=np(1-p)$. So $\mathbf{E}[Y^2]=var(Y)+\mathbf{E}[Y]^2=np(1-p)+n^2p^2$.

$$\mathbf{E}[XY]=np(1-p)+n^2p^2+n^2p^2=np(1-p)+2n^2p^2$$

(c) Are X and Y independent? Justify your answer. We have $\mathbf{E}[X]\mathbf{E}[Y]=(2np)(np)=2n^2p^2$. This is not equal to $\mathbf{E}[XY]$ so they cannot be independent.

Or consider $\mathbf{P}(X=0,Y=1)$. This is obviously 0 since $X=0$ means no successes in all $2n$ trials and so Y cannot be 1 at the same time. But $\mathbf{P}(X=1)$ and $\mathbf{P}(Y=1)$ are both nonzero. So
$\mathbf{P}(X=0,Y=1)\neq\mathbf{P}(X=1)\mathbf{P}(Y=1)$.