**Solutions - Practice problems for Exam 2**
**Math 464 - Fall 18**

**1.** Let $X$ and $Y$ be independent random variables. They both have a gamma distribution with mean 3 and variance 3.

(a) Find the joint probability density function (pdf) of $X, Y$.

**Solution:** Since they are independent it is just the product of a gamma density for $X$ and a gamma density for $Y$. For the gamma distribution, $\mu = w/\lambda$, $\sigma^2 = w/\lambda^2$. Since the mean and variance are both 3, $\lambda = 1$ and $w = 3$. So

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\Gamma(3)^2} x^2 y^2 e^{-x-y} & \text{if } x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) Express $P(3X + Y \leq 3)$ as an integral. Do not try to do the integral.

**Solution:** The region where $3x + y \leq 3, x \geq 0, y \geq 0$ is the triangle in the upper right quadrant below the line $y \leq 3 - 3x$. So we get

$$\int_0^1 \left[ \int_0^{3-3x} \frac{1}{\Gamma(3)^2} x^2 y^2 e^{-x-y} dy \right] dx$$

**2.** Let $X$ have an exponential distribution with $E[X] = 1$. Let $Y = X^2 - 2$.

(a) Find the mean and variance of $Y$.

**Solution:** First we compute some moments of $X$ for later use. The mgf for $X$ is $m(t) = 1/(1-t)$.

$$\begin{aligned} m'(t) &= \frac{1}{(1-t)^2}, \quad E[X] = m'(0) = 1, \\ m^{(2)}(t) &= \frac{2}{(1-t)^3}, \quad E[X^2] = m^{(2)}(0) = 2, \\ m^{(3)}(t) &= \frac{6}{(1-t)^4}, \quad E[X^3] = m^{(3)}(0) = 6, \\ m^{(4)}(t) &= \frac{24}{(1-t)^5}, \quad E[X^4] = m^{(4)}(0) = 24 \end{aligned}$$

Now

$$E[Y] = E[X^2] - 2 = 2 - 2 = 0$$

$$E[Y^2] = E[(X^2-2)^2] = E[X^4 - 4X^2 + 4] = 24 - 4\cdot 2 + 4 = 20$$

So $var(Y) = 20 - 0 = 20$.
(b) Find the probability density function (pdf) for $Y$.

**Solution:** We start by finding the cdf for $Y$.

$$F_Y(y) = P(Y \le y) = P(X^2 - 2 \le y) = P(X^2 \le y+2)$$
$$= P(X \le \sqrt{y+2}) = \int_0^{\sqrt{y+2}} e^{-x}\, dx = 1 - \exp(-\sqrt{y+2})$$

Take the derivative of this to get

$$f_Y(y) = \frac{1}{2}(y+2)^{-1/2}\exp(-\sqrt{y+2}), y \ge -2$$

The range for $Y$ is $[-2, \infty)$.

**3.** Let $X$ and $Y$ be continuous random variables with joint pdf

$$f_{X,Y}(x,y) = \frac{3}{2}(x^2+y^2), \quad 0 \le x \le 1, 0 \le y \le 1$$

Outside of $0 \le x \le 1, 0 \le y \le 1$, $f_{X,Y}(x,y) = 0$.
(a) Find the marginal densities of $X$ and $Y$ The marginal density of $X$ for $0 \le x \le 1$ is

$$f_X(x) = \int_0^1 \frac{3}{2}(x^2+y^2)\, dy = \frac{3}{2}[x^2 + \int_0^1 y^2\, dy] = \frac{3}{2}[x^2 + \frac{1}{3}] = \frac{1}{2} + \frac{3}{2}x^2$$

So

$$f_X(x) = \begin{cases} \frac{1}{2} + \frac{3}{2}x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The same calculation shows

$$f_Y(y) = \begin{cases} \frac{1}{2} + \frac{3}{2}y^2 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Are $X$ and $Y$ independent?

**Solution:** They are not independent since $f_{X,Y}(x,y)$ is not equal to $f_X(x)f_Y(y)$.

**4.** Let $X, Y$ be jointly continuous random variables with joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $Z = X + Y$. Compute $f_Z(z)$, the probability density function (pdf) for $Z$.

**Solution:** The range of $X, Y$ is the unit square. The range of $Z$ will be $[0, 2]$ We need to compute the cdf, $P(Z \le z) = P(X + Y \le z)$. How the line $x + y = z$ intersects the unit square depends on whether $0 \le z \le 1$ or $1 \le z \le 2$. In the first case

$$P(X + Y \le z) = \int_0^z \int_0^{z-x} 4xy \, dy \, dx$$

After some calculation this equals $\frac{1}{6}z^4$. For $1 \le z \le 2$,

$$P(X + Y \le z) = \int_0^{z-1} \int_0^1 4xy \, dy \, dx + \int_{z-1}^1 \int_0^{z-x} 4xy \, dy \, dx$$

After an unreasonable amount of calculation this equals $-\frac{1}{6}z^4 + 2z^2 - \frac{8}{3}z + 1$. So the pdf is

$$f_Z(z) = \begin{cases} \frac{2}{3}z^3, & \text{if } 0 \le z \le 1 \\ -\frac{2}{3}z^3 + 4z - \frac{8}{3}, & \text{if } 1 \le z \le 2 \\ 0, & \text{otherwise} \end{cases}$$

**5.** Let $X$ and $Y$ be independent random variables, each of which has a standard normal pdf. Let $Z = Y - X + 4$.

(a) Find the mean and variance of $Z$.

**Solution:** $E[Z] = E[Y] - E[X] + 4 = 4$. $var(Z) = var(Y) + var(-X) = var(Y) + var(X) = 1 + 1 = 2$.

(b) Find the probability density function (pdf) of $Z$. Hint: this can be done with very little computation.

**Solution:** It is easy to show that $-X$ is also a standard normal. The sum of independent normal random variables is normal, and adding a constant to a normal random variable gives another normal random variable. So $Z$ is normal. Part (a) tells us its mean and variance. Another way to see this is to look at the mgf. Since $X$ and $Y$ are independent, functions of them are independent. So

$$\begin{aligned} M_Z(t) &= E[\exp(t(Y-X+4))] = e^{4t}E[\exp(tY)]E[\exp(-tX)] \\ &= \exp(4t+\frac{1}{2}t^2+\frac{1}{2}(-t)^2) = \exp(4t+t^2) \end{aligned}$$

which is mgf of a normal with mean 4 and variance 2. So

$$f_Z(z) = \frac{1}{\sqrt{4\pi}}\exp(\frac{1}{4}(x-4)^2), \quad -\infty < z < \infty$$

**6.** Random variables $X$ and $Y$ have joint cumulative distribution function (cdf)

$$F_{X,Y}(x,y) = \begin{cases} [\frac{1}{\pi}\tan^{-1}(x)+c](1-e^{-y}), & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases}$$

where $c$ is some constant.
(a) Are $X$ and $Y$ independent?

**Solution:** Yes, the joint cdf factors into a function of $x$ times a function of $y$, so they are independent.

(b) Find the value of $c$.

**Solution:**

$$\lim_{x,y\to\infty} F(x,y) = \frac{1}{\pi}\frac{\pi}{2} + c = \frac{1}{2} + c$$

This must equal 1, so $c = 1/2$.
(c) Find the joint probability density function (pdf) for $X, Y$.

**Solution:** We take the second order partial derivative of $F_{X,Y}(x,y)$ with respect to $x$ and $y$. This gives

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}\frac{1}{1+x^2}e^{-y}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases}$$

Note that $X, Y$ are independent. $X$ has the Cauchy distribution, and $Y$ is exponential with $\lambda = 1$.

**7.** A RV $X$ has a Laplace distribution if its pdf is

$$f_X(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, \quad -\infty < x < \infty$$

where $\lambda > 0$ is a parameter.
(a) Compute the moment generating function of $X$.
**Solution:**

$$\begin{aligned}
M_X(t) &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{tx}\, e^{-\lambda|x|}\, dx \\
&= \frac{\lambda}{2}\int_{-\infty}^{0} e^{tx+\lambda x}\, dx + \frac{\lambda}{2}\int_{0}^{\infty} e^{tx-\lambda x}\, dx \\
&= \frac{\lambda}{2}\frac{1}{t+\lambda} e^{tx+\lambda x}|_{x=-\infty}^{x=0} + \frac{\lambda}{2}\frac{1}{t-\lambda} e^{tx-\lambda x}|_{x=0}^{x=\infty} \\
&= \frac{\lambda}{2}\frac{1}{t+\lambda} - \frac{\lambda}{2}\frac{1}{t-\lambda} \\
&= \frac{\lambda^2}{\lambda^2 - t^2}
\end{aligned}$$

(b) Find the mean and variance of $X$.
**Solution:**

$$M'(t) = \frac{2t\lambda^2}{(\lambda^2 - t^2)^2}$$

So $M'(0) = 0$. So the mean is zero.

$$M''(t) = \frac{2\lambda^2}{(\lambda^2 - t^2)^2} + \frac{8t^2\lambda^2}{(\lambda^2 - t^2)^4}$$

So $M''(0) = 2/\lambda^2$. Since the mean is zero, the variance is $2/\lambda^2$.

**8.** Let $X_j$ be a sequence of independent, identically distributed random variables. Their common density is

$$f(x) = 4xe^{-2x}, x \geq 0$$

(It is zero for $x < 0$.) Let

$$\overline{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j$$

**(a)** Find the mean and variance of $\overline{X}_n$. Hint: this density is in our catalog.
**Solution:** The $X_j$ have a gamma distribution with $w = 2$ and $\lambda = 2$. So $E[X_j] = 1$ and $var(X_j) = 1/2$. So the mean of $\overline{X}_n$ is 1 and its variance is $\frac{1}{2n}$.
**(b)** For $n = 1000$, the probability that $\overline{X}_{1000}$ is in $[1, 1.1]$ is approximately given by

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Find $a$ and $b$.
**Solution:**

$$\begin{aligned} P(1 \leq \overline{X}_{1000} \leq 1.1) &= P(0 \leq \frac{\overline{X}_{1000} - 1}{\frac{1}{\sqrt{2n}}} \leq \frac{1.1 - 1}{\frac{1}{\sqrt{2n}}}) \\ &= P(0 \leq Z \leq 0.1\sqrt{2n}) \end{aligned}$$

So $a = 0$ and $b = 0.1\sqrt{2n} = 2\sqrt{5}$.

**9.** A random number generator produces random real numbers that are uniformly distributed between $-1$ and 1. I call it $n$ times and let $\overline{X}$ be the average of the $n$ numbers I get. (So $\overline{X}$ is the sum of the $n$ numbers divided by $n$.) As $n$ goes to $\infty$, $P(|\overline{X}| \geq 0.01)$ goes to zero. Find $n$ so that this probability is approximately 0.05.

For a standard normal RV $Z$, we have $P(Z \leq -2.576) = 0.005$, $P(Z \leq -2.326) = 0.01$, $P(Z \leq -1.960) = 0.025$, $P(Z \leq -1.649) = 0.05$.
**Solution:** The mean of $X_i$ is 0 and the variance is $1/3$. So the mean of $\overline{X}$ is 0 and the variance is $\frac{1}{3n}$. So

$$Z = \frac{\overline{X} - 0}{\sqrt{\frac{1}{3n}}} = \sqrt{3n}\,\overline{X}$$

So we want $P(|Z| \geq \sqrt{3n}\,0.01) = 0.05$. So $P(Z \leq -\sqrt{3n}\,0.01) = 0.025$. So $\sqrt{3n}\,0.01 = 1.96$. And so $n = (196)^2/3 = 12,805$