Final Exam Review problems - Math 464 - Fall 2018

1. An urn has 1 red ball and 2 green balls. I draw a ball. If it is red, I put it back in the urn. If it is green I do not put it back. Then I draw a second ball.

(a) Find the probability the second ball is red.
(b) Someone does this experiment and tells us he got a red ball on the second draw. What is the probability he got a red ball on the first draw?
(c) Are the events "first ball is red" and "second ball is red" independent? Explain your reasoning.

2. Let X have the gamma distribution with $\lambda = 1/2$ and $w = 1/2$. Find the probability density function (pdf) of $Y = \sqrt{X}$.

3. I roll two tetrahedral (four-sided) dice. Let X be the number of dice that show an odd number, Y the number that show an even number. So both X and Y only take on the values $0, 1, 2$.

(a) Find the joint probability mass function (pmf) for X, Y. You can give your answer in the form of a table.

(b) Compute $P(Y \geq X)$.

(c) Are X and Y independent? Justify your answer. Note: you can do (c) without doing (a) or (b).

4. Let X and Y be independent continuous random variables. X has a uniform distribution on $[0, 1]$ and Y has an exponential distribution with $E[Y] = 1$. Let $Z = Y - X$. Compute $P(Z \geq 0)$.

5. I roll a fair six-sided die until I get a 1 or a 2. Let X be the number of rolls this takes. Then I continue rolling the die until I get that same number a second time. Here are some possible outcomes:
$3, 4, 5, 4, 1, 2, 3, 4, 1 \quad OR \quad 6, 6, 3, 2, 1, 2 \quad OR \quad 5, 6, 6, 3, 1, 3, 3, 2, 1 \quad OR \quad 2, 2$
Let N be the total number of rolls.

(a) Find the mean and variance of X.

(b) Find the mean and variance of N. Hint : you can write N as $X + Y$ where X and Y are independent, relatively simple RV's.

6. The joint probability density function (pdf) of two continuous random variables X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 4\, e^{-2x-y}, & \text{if } x \geq 0, y \geq 0, y \geq 2x \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal pdf of X.

(b) Find $P(Y \geq 1 | X = x)$, i.e., the probability that $Y \geq 1$ given that $X = x$.

7. Let X and Y be independent random variables. Both are exponential with $\lambda = 2$. Let

$$U = \frac{1}{2}(X + Y), \qquad W = Y^2$$

(a) What is the range of (U, W)?

(b) Find the joint pdf for U, W.

8. Let X_j be a sequence of independent, identically distributed random variables. Their common density is

$$f(x) = 4xe^{-2x}, x \geq 0$$

(It is zero for $x < 0$.) Let

$$\overline{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j$$

(a) Find the mean and variance of $\overline{X}_n$. Hint: this density is in our catalog.

(b) For $n = 1000$, the probability that $\overline{X}_{1000}$ is in $[1, 1.1]$ is approximately given by

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Find a and b.

9. The time it takes to solve a problem is exponentially distributed with $\lambda = 1/2$. Two students named Fred and Wilma begin working on the problem at the same time. (Assume that the students are working independently).

(a) Find the probability that at least one of the students has solved the problem at the end of 2 minutes.

(b) Find the probability that Fred takes at least twice as long as Wilma to solve the problem.

10. A mystery random number generator produces random real numbers that are uniformly distributed between a and $a + 1$. I don't know what a is.

(a is not random, it is just unknown.) I call the random number generator n times and let $\overline{X}_n$ be the average of the n numbers I get. (So $\overline{X}_n$ is the sum of the n numbers divided by n.) As n goes to ∞, $\overline{X}_n$ converges to $a + 1/2$. Find the smallest n so that the probability that $\overline{X}_n$ is within 0.01 of $a + 1/2$ is approximately 0.95. For a standard normal RV Z, we have $P(Z \leq -2.576) = 0.005$,
$P(Z \leq -2.326) = 0.01$, $P(Z \leq -1.960) = 0.025$, $P(Z \leq -1.649) = 0.05$.

11. X and Y are discrete random variables with joint pmf

$$f_{X,Y}(x, y) = \frac{\lambda^{x+y}\, e^{-2\lambda}}{x!\, y!}$$

where x and y both take on the values $0, 1, 2, 3, \cdots$, and λ is a positive parameter.

(a) Find the marginal pmf's of X and Y.

(b) Find $E[XY]$.

(c) Let $Z = X + Y$. Find $E[Z|X = x]$.

(d) Let $Z = X + Y$. Find the moment generating function of Z. Z is in our catalog. What is it?

12. I roll a fair six-sided die until I get an even number. Call it N. Then I roll it some more until I get a number at least as big as N. Let X be the total number of rolls. Find the mean of X.

13. A continuous random variable Z has range equal to the entire real line with pdf $f_Z(z) = e^{-|z|}/2$.

(a) Find the moment generating function (mgf) of Z.

(b) Let X and Y be independent and identically distributed random variables. They have the same mgf, call it $M(t)$. Let $W = X - Y$. Show that the mgf of W satifies $M_W(t) = M(t)M(-t)$.

(c) If we take X and Y to be independent and identically distributed with one of the continuous pdf's in our catalog, then $W = X - Y$ will have the same pdf as Z. For which pdf in the catalog is this true? You should justify your answer. Hint : $1 - t^2 = (1 - t)(1 + t)$.