1 Moment generating functions - supplement to chap 1

The moment generating function (mgf) of a random variable X is

$$M_X(t) = E[e^{tX}] \tag{1}$$

For most random variables this will exist at least for t in some interval containing the origin. The mgf is a computational tool. By taking derivatives and evaluating them at $t = 0$ you can compute moments:

$$M'(0) = E[X], \quad M''(0) = E[X^2], \quad M^{(k)}(0) = E[X^k] \tag{2}$$

If $Y = aX + b$ for constants a and b, then

$$M_Y(t) = e^{bt} M_X(at) \tag{3}$$

If $X_1, X_2, \cdots X_n$ are independent and $Y = X_1 + \cdots X_n$, then

$$M_Y(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t) \tag{4}$$

1.1 Discrete distributions

Bernouilli distribution: This is a random variable X that only equals 0 and 1. The parameter p is $P(X = 1)$.

$$E[X] = p, \quad Var(X) = p(1-p), \quad M(t) = (1-p) + pe^t \tag{5}$$

Binomial distribution: Flip a coin n times, X is the number of heads, p is the probability of heads.

$$f(x|n,p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n \tag{6}$$

$$E[X] = np, \quad Var(X) = np(1-p), \quad M(t) = [(1-p) + pe^t]^n \tag{7}$$

Note that the binomial random variable is the sum of n independent Bernoulli random variables with the same p.

Poisson: For $\lambda > 0$,

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots \tag{8}$$

$$E[X] = \lambda, \quad Var(X) = \lambda, \quad M(t) = \exp(\lambda(e^t - 1)) \tag{9}$$

Geometric: Flip a coin until we get heads for the first time. X is the number of tails we get before this first heads.

$$f(x|p) = p(1-p)^x, \quad x = 0, 1, 2, \cdots \tag{10}$$

$$E[X] = \frac{1-p}{p}, \quad Var(X) = \frac{1-p}{p^2}, \quad M(t) = \frac{p}{1-(1-p)e^t} \tag{11}$$

Warning: some people define X to be the total number of flips including the one that gave you the first head.

Negative binomial: Flip a coin until we get heads for the kth time. X is the number of flips including the flip on which the kth head happened.

$$E[X] = \frac{k(1-p)}{p}, \quad Var(X) = \frac{k(1-p)}{p^2}, \quad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^k \tag{12}$$

Warning: some people define X to be the total number of flips including the ones that gave you the first k heads.

1.2 Continuous distributions

Normal: For $\sigma > 0$ and any μ,

$$f(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty \tag{13}$$

$$E[X] = \mu, \quad Var(X) = \sigma^2, \quad M(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2) \tag{14}$$

Exponential: For $\lambda > 0$,

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0 \tag{15}$$

$$E[X] = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}, \quad M(t) = \frac{\lambda}{\lambda - t} \tag{16}$$

Gamma: For $\alpha, \beta > 0$,

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta x}, \quad x \geq 0 \tag{17}$$

If α is an integer, $\Gamma(\alpha) = (\alpha - 1)!$.

$$E[X] = \frac{\alpha}{\beta}, \quad Var(X) = \frac{\alpha}{\beta^2}, \quad M(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \tag{18}$$

This shows that the sum of k independent exponential random variables with parameter λ has a gamma distribution with $\alpha = k$ and $\beta = \lambda$.
Warning: Some people parameterize the gamma distribution differently. Their β is my $1/\beta$.

Chi-squared or χ^2 : Let $Z_1, Z_2, \cdots, Z_n$ be independent standard normal RV's. Let

$$X = \sum_{i=1}^n Z_i^2 \tag{19}$$

Then X has the chi-squared distribution with n degrees of freedom. It can be shown that this is the gamma distribution with $\alpha = n/2$ and $\beta = 1/2$. So the pdf is

$$f(x|n) = \frac{1}{2^{n/2}\Gamma(n/2)}\, x^{n/2-1}\, e^{-x/2}, \quad x \geq 0 \tag{20}$$

$$E[X] = n, \quad Var(X) = 2n, \quad M(t) = \left(\frac{1}{1-2t}\right)^{n/2} \tag{21}$$

Note that the sum of independent chi-squared is again with chi-squared with the number of degree of freedom adding.

Student's t: Let U and V be independent random variables. U has a standard normal distribution, and V has a chi-square distribution with n degrees of freedom. Let

$$T = \frac{U}{\sqrt{V/n}} \tag{22}$$

The distribution of T is called Student's t distribution (or just the t distribution) with n degrees of freedom. The pdf is

$$f(x|n) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{(n\pi)^{1/2}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \quad -\infty < x < \infty \tag{23}$$

The mgf is not defined.