1 Hypothesis Testing - Uniformly Most Powerful Tests

We give the definition of a uniformly most powerful test in a general setting which includes one-sided and two-sided tests. We take the null hypothesis to be

$$H_0: \quad \theta \in \Omega_0$$

and the alternative to be

$$H_1: \quad \theta \in \Omega_1$$

We write the power function as $Pow(\theta, d)$ to make its dependence on the decision function explicit.

Definition: A decision function d^* is a uniformly most powerful (UMP) decision function (or test) at significance level α_0 if
(1) $Pow(\theta, d^*) \leq \alpha_0, \quad \forall \theta \in \Omega_0$
(2) For every decision function d which satisfies (1), we have $Pow(\theta, d) \leq Pow(\theta, d^*), \quad \forall \theta \in \Omega_1$.

Do UMP tests ever exist? If the alternative hypothesis is one-sided then they do for certain distributions and statistics. We proceed by defining the needed property on the population distribution and the statistic.

Definition: Let $T = t(X_1, X_2, \cdots, X_n)$ be a statistic. Let $f(x_1, x_2, \cdots, x_n|\theta)$ be the joint density of the random sample. We say that $f(x_1, x_2, \cdots, x_n|\theta)$ has a *monotone likelihood ratio in the statistic* T if for all $\theta_1 < \theta_2$ the ratio

$$\frac{f(x_1, \cdots, x_n|\theta_2)}{f(x_1, \cdots, x_n|\theta_1)}$$

depends on $x_1, \cdots, x_n$ only through $t(x_1, \cdots, x_n)$ and the ratio is an increasing function of $t(x_1, \cdots, x_n)$.

Example: Consider a Bernoulli distribution for the population, i.e., we are looking at a population proportion. So each $X_i = 0, 1$ and $p = P(X_i = 1)$. The joint density is

$$f(x_1, \cdots, x_n|p) = p^{n\bar{x}}(1-p)^{n-n\bar{x}}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Let $p_1 < p_2$. We have

$$\frac{f(x_1, \cdots, x_n|p_2)}{f(x_1, \cdots, x_n|p_1)} = \left[\frac{p_2(1-p_1)}{p_1(1-p_2)}\right]^{n\bar{x}} \left[\frac{1-p_2}{1-p_1}\right]^{n}$$

So the ratio depends on the sample only through the sample mean $\bar{x}$ and it is an increasing function of $\bar{x}$. (It is an easy algebra exercise to check that if $p_2 > p_1$ then $p_2(1-p_1)/(p_1(1-p_2)) > 1$.)

Example: Now consider a normal population with unknown mean μ and known variance σ^2. So the joint density is

$$f(x_1, \cdots, x_n|\mu) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp(-\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2)$$

Now let $\mu_1 < \mu_2$. A little algebra shows

$$\frac{f(x_1, \cdots, x_n|\mu_2)}{f(x_1, \cdots, x_n|\mu_1)} = \exp(\frac{n}{\sigma^2}\bar{x}(\mu_2 - \mu_1) + \frac{(\mu_1^2 - \mu_2^2)n}{2\sigma^2})$$

So the ratio depends on $x_1, x_2, \cdots, x_n$ only through $\bar{x}$, and the ratio is an increasing function of $\bar{x}$.

Theorem 1. *Suppose $f(x_1, \cdots, x_n|\theta)$ has a monotone likelihood ratio in the statistic $T = t(X_1, \cdots, X_n)$. Consider hypothesis testing with alternative hypothesis $H_a : \theta > \theta_1$, and null hypothesis $H_0 : \theta \leq \theta_0$ or $H_0 : \theta = \theta_0$. Let α_0, c be constants such that $P(T \geq c) = \alpha_0$. Then the test that rejects the null hypothesis if $T \geq c$ is a UMP test at significance level α_0.*

Example: We continue the example of a normal population with known variance and unknown mean. We saw that the likelihood ratio is monotone in the sample mean. So if we reject the null hypothesis when $\bar{X}_n \geq c$, this will be a UMP test with significance level $\alpha = P(\bar{X}_n \geq c|\mu_0)$. Given a desired significance level α, we choose c so this equation holds. Then the theorem tells us we have a UMP test. So for every $\mu > \mu_0$, our test makes $Pow(\mu)$ as large as possible.

Example: We continue the example of a Bernoulli distribution for the population (population proportion). To be concrete, suppose the null hypothesis is $p \leq 0.1$ and the alternative is $p > 0.1$. We have a random sample of size $n = 20$. Let $\bar{X}$ be the sample proportion. By what we've already done, the test that reject the null hypothesis when $\bar{X} \geq c$ will be a UMP test. We want to choose c so that $P(\bar{X} \geq c) = \alpha_0$. However, $\bar{X}$ is a discrete RV (it can only be $0/20, 1/20, 2/20, \cdots, 19/20, 1$), so this is not possible. Suppose we want a significance level of 0.005 Using your favorite software (or a table of the binomial distibution) we find that $P(\bar{x} \geq 6/20|p = 0.1) = 0.0113$ and $P(\bar{x} \geq 7/20|p = 0.1) = 0.0024$. So we must take $c = 7/20$. Then the test that rejects the null if $\bar{x} \geq 7/20$ is a UMP test at significance level 0.005.

What about two-sided alternatives? It can be shown that there is no UMP test in this setting.