Math 520a - Final take home exam - version 1

Do 5 out of the 6 problems. Do not turn in more than 5.

You may work on the exam for one continuous week during the period Mon, Dec 7 to Fri, Dec 18. The exam is due at 3 pm on Fri, Dec 18 at the latest. Late papers will only be accepted in the case of illness. You may consult the textbook, your class notes and other books as long as you are not looking up the actual problem. You may not consult other people or the web. You can ask me to clarify the problem statement, but I will not give hints.

1. Let $f(z)$ be entire. Prove that f has finite order if and only if f' has finite order and that when they have finite order their orders are the same.

2. Consider the entire function $1/\Gamma(z)$.

(a) Show it does not satisfy

$$|\frac{1}{\Gamma(z)}| \leq A \exp(B|z|)$$

for any constants A, B. Hint: look at the points $-n - 1/2$ where n is a positive integer.

(b) Show there is no entire function satisfying a bound of the above form with simple zeros at $0, -1, -2, -3, \cdots$ and no other zeroes.

3. Suppose there is an entire function $f(z)$ and a polynomial $p(z)$ such that $p(f(z)) = e^z$ for all z. Prove that $p(z)$ can only have one root.

4. Prove that for all $z \in \mathbb{C}$

$$\cos(\frac{\pi z}{2}) = \prod_{n=0}^{\infty} \left[1 - \frac{z^2}{(2n+1)^2} \right]$$

5. (a) Prove that for $R < 1$ there is a constant $c(R)$ such for all complex a with $|a| < 1 - R$ and all f which are analytic on the unit disc $\mathbb{D}$, we have

$$|f(a)| \leq c(R) \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})| \, r dr d\theta$$

(b) Let f_n be analytic on $\mathbb{D}$ and f continuous on $\mathbb{D}$. Suppose that f_n converges to f in $L^1(\mathbb{D})$ meaning that

$$\int_0^{2\pi} \int_0^1 |f_n(re^{i\theta}) - f(re^{i\theta})| \, r dr d\theta \to 0$$

Prove that f is analytic.

6. Prove the following theorem. We have proved most of the various implications needed in class. For many implications you can just cite a theorem we have proved. For example, to prove (a) implies (e) you can just cite the Riemann mapping theorem. Hint: just what property of Ω did we need in the proof of the Riemann mapping theorem ?

Theorem: The following are equivalent for a connected, open set $\Omega \subset \mathbb{C}$.

(a) Ω is simply connected, i.e., every closed curve is homotopic to a point.

(b) For every analytic function f on Ω and every closed contour γ in Ω, we have

$$\int_\gamma f(z)\,dz = 0$$

(c) For every analytic function f on Ω there is an analytic function F on Ω such that $F' = f$.

(d) For every analytic function f on Ω which does not vanish on Ω there is an analytic function g on Ω such that $e^g = f$.

(e) Either $\Omega = \mathbb{C}$ or there is a conformal map from Ω onto the unit disc.