Math 520a - Final take home exam - solutions

1. Let $f(z)$ be entire. Prove that f has finite order if and only if f' has finite order and that when they have finite order their orders are the same.

Solution: Suppose that f satisfies

$$|f(z)| \leq A \exp(B|z|^\sigma)$$

By the Cauchy integral formula,

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^2}, dw$$

where we take γ to be a circle of radius 1 centered at z. On this contour, $|f(w)| \leq A \exp(B(|z|+1)^\sigma)$ and $|w| \geq |z| - 1$. So

$$|f'(z)| \leq A \frac{\exp(B(|z|+1)^\sigma)}{(|z|-1)^2} \leq A' \exp(B'|z|^\sigma)$$

for some constants A', B', with the same σ as in the bound on f. It follows that if f has order ρ, then f' has order less than or equal to ρ.

Now suppose f' satisfies

$$|f'(z)| \leq A \exp(B|z|^\sigma)$$

We have

$$f(z) = f(0) + \int_0^z f'(w)\, dw$$

So

$$|f(z)| \leq |f(0)| + |z| A \exp(B|z|^\sigma)$$

For any $\epsilon > 0$ there are constants A', B' such that the above is

$$\leq A' \exp(B'|z|^{\sigma+\epsilon})$$

It follows that if f' has order ρ, then f has order less than or equal to ρ.

2. Consider the entire function $1/\Gamma(z)$.

(a) Show it does not satisfy

$$|\frac{1}{\Gamma(z)}| \leq A \exp(B|z|)$$

for any constants A, B. Hint: look at the points $-n-1/2$ where n is a positive integer.

Solution: We showed in class that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Take $z = -n-1/2$. Then $\sin(\pi(-n-1/2)) = \pm 1$. So

$$\left|\frac{1}{\Gamma(-n-1/2)}\right| = \frac{1}{\pi}|\Gamma(1-(-n-1/2))| = \frac{1}{\pi}|\Gamma(n+3/2))|$$
$$= \frac{1}{\pi}(n+1/2)(n-1/2)\cdots 3/2|\Gamma(1/2))| \geq \frac{n!}{\pi}|\Gamma(1/2))| = \frac{n!}{\sqrt{\pi}}$$

We have $|z| = n+1/2$, and for any constants A, B, $n!$ is eventually greater than $A\exp(B(n+1/2))$.

(b) Show there is no entire function satisfying a bound of the above form with simple zeros at $0, -1, -2, -3, \cdots$ and no other zeroes.

Solution: Let $f(z)$ be such a function. Then it has order less than or equal to 1. By the Hadamard factorization theorem,

$$f(z) = \exp(az+b)\prod_{n=1}^{\infty} E_1(-z/n)$$

Recall that $1/\Gamma(z)$ is also of this form with $a = \gamma$ and $b = 0$. So $1/\Gamma(z) = \exp(\gamma z - az - b)f(z)$. This implies $1/\Gamma(z)$ satisfies a bound of the form $A\exp(B|z|)$ and we know from part (a) that it does not.

3. Suppose there is an entire function $f(z)$ and a polynomial $p(z)$ such that $p(f(z)) = e^z$ for all z. Prove that $p(z)$ can only have one root.

Solution: Since e^z is never zero, $f(z)$ cannot equal a root of $p(z)$. So $f(\mathbb{C})$ cannot contain any of the roots. By the little Picard theorem, $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C}$ minus a single point. So there can only be one root.

4. Prove that for all $z \in \mathbb{C}$

$$\cos(\frac{\pi z}{2}) = \prod_{n=0}^{\infty} \left[1 - \frac{z^2}{(2n+1)^2}\right]$$

Solution: $\cos(\frac{\pi z}{2})$ is an entire function and since $|\cos(\frac{\pi z}{2})| \leq \exp(\pi|z|/2)$, it has finite order with order ≤ 1. It has zeroes at $2n+1$ where n is an integer. We can also label the zeroes as $\pm(2n+1)$ with $n = 0, 1, 2, \cdots$. So the Hadamard factorization theorem says

$$\cos(\frac{\pi z}{2}) = e^{az+b} \prod_{n=0}^{\infty} [E_1(\frac{z}{(2n+1)}) E_1(-\frac{z}{(2n+1)})]$$

(The product converges absolutely, so we can order it any way we want.) Since

$$[E_1(\frac{z}{(2n+1)}) E_1(-\frac{z}{(2n+1)})]$$
$$= (1 - \frac{z}{2n+1}) \exp(\frac{z}{2n+1})(1 + \frac{z}{(2n+1)}) \exp(-\frac{z}{2n+1}) = (1 - \frac{z^2}{(2n+1)^2})$$

to finish the proof we need to show $a = 0, b = 0$. Evaluating at $z = 0$ shows $e^b = 1$. The evenness of cos and of the infinite product shows $a = 0$.

5. (a) Prove that for $R < 1$ there is a constant $c(R)$ such for all complex a with $|a| < 1 - R$ and all f which are analytic on the unit disc $\mathbb{D}$, we have

$$|f(a)| \leq c(R) \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})| \, r dr d\theta$$

So

Solution: Since $|a| + R < 1$, if $r \leq R$, then the circle of radius r centered at a is contained in $\mathbb{D}$. By Cauchy's integral formula

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z} dz \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \end{aligned}$$

So

$$|f(a)| \quad \leq \quad \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|d\theta$$

Multiply both sides by r and integrate from 0 to R.

$$|f(a)| \int_0^R r\,dr \leq \frac{1}{2\pi}\int_0^R \int_0^{2\pi} |f(a+re^{i\theta})| r\,d\theta dr$$

This proves (a) with $c(R) = \frac{2}{R^2\pi}$.

(b) Let f_n be analytic on $\mathbb{D}$ and f continuous on $\mathbb{D}$. Suppose that f_n converges to f in $L^1(\mathbb{D})$ meaning that

$$\int_0^{2\pi}\int_0^1 |f_n(re^{i\theta}) - f(re^{i\theta})|^2\, rdrd\theta \to 0$$

Prove that f is analytic.

Solution: It suffice to show that f_n converges to f uniformly on the disc $|z| < 1-\delta$ for all $\delta > 0$. (Uniform limits of analytic functions are analytic.) Fix such an $\delta > 0$. For any constant c, $f_n(z) - c$ is an analytic function of z and so part (a) implies that for any for $a \in D_{1-\delta}(0)$ and any r such that $r < \delta/2$,

$$|f_n(a) - c| \leq c(r)\int_0^{2\pi}\int_0^r |f_n(a+re^{i\theta}) - c|\, rdrd\theta$$

Take $c = f(a)$ and we have

$$|f_n(a) - f(a)| \leq c(r)\int_0^{2\pi}\int_0^r |f_n(a+re^{i\theta}) - f(a)|\, rdrd\theta$$

By triangle inequality this is

$$\begin{aligned} &\leq \quad c(r)\int_0^{2\pi}\int_0^r |f_n(a+re^{i\theta}) - f(a+re^{i\theta})|\, rdrd\theta \\ &+ \quad c(r)\int_0^{2\pi}\int_0^r |f(a+re^{i\theta}) - f(a)|\, rdrd\theta \end{aligned}$$

The first term may be bounded by $c(r)||f_n - f||_1$ since the region of integration is contained inside the unit disc. The second term may be bounded by

$$c(r)\pi r^2 \sup_{z:|z-a|\leq r} |f(z) - f(a)| \tag{1}$$

Note that $c(r)\pi r^2 = 2$. And the requirement that $r \leq \delta/2$ means that z in the above sup lies in $D_{1-\delta/2}(0)$. Since f is uniformly continuous on the compact set $\overline{D_{1-\delta/2}(0)}$, the above sup goes to 0 as $r \to 0$. We have to be a bit careful with the order in which we choose things. Let $\epsilon > 0$. Pick r so that (1) is $< \epsilon/2$. Then pick N so that $c(r)||f_n - f||_1 < \epsilon/2$. Then putting this all together gives $|f_n(a) - f(a)| < \epsilon$ for $|z| < R$ and $n > N$. This shows f_n converges to f uniformly on $D_r(0)$.

6. Prove the following theorem. We have proved most of the various implications needed in class. For many implications you can just cite a theorem we have proved. For example, to prove (a) implies (e) you can just cite the Riemann mapping theorem. Hint: just what property of Ω did we need in the proof of the Riemann mapping theorem ?

Theorem: The following are equivalent for a connected open set $\Omega \subset \mathbb{C}$.

(a) Ω is simply connected, i.e., every closed curve is homotopic to a point.

(b) For every analytic function f on Ω and every closed contour γ in Ω, we have

$$\int_\gamma f(z)\, dz = 0$$

(c) For every analytic function f on Ω there is an analytic function F on Ω such that $F' = f$.

(d) For every analytic function f on Ω which does not vanish on Ω there is an analytic function g on Ω such that $e^g = f$.

(e) Either $\Omega = \mathbb{C}$ or there is a conformal map from Ω onto the unit disc.

Solution:

(a) $\Rightarrow$ (b) : This is Cauchy's theorem.

(b) $\Rightarrow$ (c) : We proved this in class. The idea is that you fix a point z_0 in Ω and define

$$F(z) = \int_\gamma f(w)\, dw$$

where γ is any contour from z_0 to z. Property (b) implies that the integral does not depend on the choice of contour. It is then routine to show $F' = f$.

(c) $\Rightarrow$ (d) : We proved this in class. Since f is never zero, f'/f is analytic on Ω. So there is an F with $F' = f'/f$. The the derivative of $e^{-F} f$ is zero. So $e^{-F} = c$. Let $g(z) = F(z) + \ln(c)$.

(d) $\Rightarrow$ (e) : Riemann's theorem says that (a) $\Rightarrow$ (e). But if you look at the proof you see that the only property we needed for Ω was (d) and the existence of square roots of functions on Ω which never vanish which follows immediately from (d).

(e) $\Rightarrow$ (a) : A conformal map is a homeomorphism. So Ω has whatever topological properties the unit disc does. In particular, Ω is simply connected.

We have proved (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), which proves the five properties are equivalent.