Math 520a - Homework 0

You do not have to turn in the first 6 problems.

1. For $A \subset \mathbb{C}$, define $int(A) = \{z : \exists \epsilon > 0\, such\, that\, D_\epsilon(z) \subset A\}$.
(a) Prove $int(A)$ is open.
(b) Prove that if U is open and $U \subset A$, then $U \subset int(A)$.

2. Define $\bar{A} = \{z : \exists z_n \in A\, such\, that\, z_n \to z\}$.
(a) Prove $\bar{A}$ is closed.
(b) Prove that if F is closed and $A \subset F$, then $\bar{A} \subset F$.
(c) Prove $\bar{A} = (int(A^c))^c$.

3. For $A \subset \mathbb{C}$ define $\partial A = \bar{A} \setminus int(A)$. Prove that

$$\partial A = \{z : \exists z_n \in A\, with\, z_n \to z,\, and\, \exists w_n \notin A\, with\, w_n \to z\}$$

4. *The complex exponential:* One way to define e^z for complex z is by its power series. Here is another. Letting $z = x + iy$, we should have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i\sin(y)]$$

So we can define e^{x+iy} to be the complex valued function whose real part is $u(x+iy) = e^x \cos(y)$ and whose imaginary part is $v(x+iy) = e^x \sin(y)$.
(a) Prove this is an entire function and it satisifes the differential equation $(e^z)' = e^z$.
(b) Let a be real and let C be the vertical line given by $Re(z) = a$. What is the image of C under the map e^z ?

5. Let Ω be the complex plane with the ray $(-\infty, 0]$ on the real axis removed:

$$\Omega = \mathbb{C} \setminus \{z : Im(z) = 0, Re(z) \leq 0\}$$

Any $z \in \Omega$ can be written uniquely as $re^{i\theta}$ with $-\pi < \theta < \pi$, $r > 0$. Define $\ln(z)$ to be $\ln(r) + i\theta$. Prove that $\ln(z)$ is analytic on Ω and that $e^{\ln(z)} = z$.

6. Define Ω as in the previous problem. The square root can be defined by $\sqrt{z} = \exp(\ln(z)/2)$. Let $\mathbb{H}$ be the upper half of the complex plane (not including the real axis).
(a) What is the image of Ω and of $\mathbb{H}$ under the map $\sqrt{z}$?

The following two problems should be turned in.

7. Let $f(z)$ be defined on a neighborhood of z_0. Suppose there is a complex number w such that for all angles θ,

$$\lim_{r \to 0^+} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}} = w$$

Does it follow that f is complex differentiable at z_0? Prove that it does or give a counterexample. In the above r goes to 0 only through positive real numbers.

8. Let $f(z) = \sqrt{1 - z^2}$ with $\sqrt{\cdots}$ defined as in previous problem. What is the image of the upper half plane $\mathbb{H}$ under f ?