Math 520a - Homework 1

1. Let f be analytic on a region Ω (a connected open set). Prove that in each of the following cases f is a constant.
(a) $f(\Omega)$ is a subset of the real line.
(b) $f(\Omega)$ is a subset of some line.
(c) $f(\Omega)$ is a subset of some circle.
2. Let Ω be an open set and f analytic on Ω. Define $\overline{\Omega} = \{\overline{z} : z \in \Omega\}$. Define

$$g(z) = \overline{f(\overline{z})}$$

Prove that g is analytic on $\overline{\Omega}$.
3. Let γ be the square with corners at $1+i, -1+i, -1-i, 1-i$ traversed in the counterclockwise direction. Compute

$$\int_\gamma \bar{z}\, dz$$

4. Let γ be a curve with bounded variation which is not necessarily piecewise smooth. Let f be continuous on γ. Let γ^- be γ with the opposite orientation, i.e., traversed in the opposite direction. Prove that

$$\int_{\gamma^-} f(z)\, dz = -\int_\gamma f(z)\, dz$$

Note that since γ is not assumed to be smooth, you will have to use the definition of the integral I gave in class.
5. Let f be analytic on $\mathbb{C} \setminus \{0\}$ and suppose that

$$\int_R f(z) dz = 0$$

for all rectangles R that do not pass through 0. (R can enclose 0.) Prove that f has a primitive on $\mathbb{C} \setminus \{0\}$. You should only use results up to p. 41 in the book.
6. Book number 7 on p. 26 on "Blaschke factors."
7. Suppose a_n is a sequence of complex numbers for which $\rho = \limsup_{n\to\infty} |a_n|^{1/n}$ is not 0 or infinity. Consider

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^{-n}$$

(a) Prove that this series converges absolutely on $\{z : |z| > \rho\}$ and $f(z)$ is analytic on this set.
(b) Define

$$f_N(z) = \sum_{n=0}^{N} a_n z^{-n}$$

Prove that f_N converges uniformly to f on $\{z : |z| = r\}$ for $r > \rho$.
(c) Use (b) to prove that

$$\int_\gamma f(z)dz = \lim_{N\to\infty} \int_\gamma f_N(z)dz = \sum_{n=0}^{\infty} a_n \int_\gamma z^{-n}dz = 2\pi i a_1$$

where γ is a circle of radius r centered at the origin. Can you generalize this to other contours?

Following problems should not be turned in .

8. Problems 5 and 6 on pp. 25-26 of the book.