Math 520a - Homework 1

1. Let f be analytic on a region Ω (a connected open set). Prove that in each of the following cases f is a constant.
(a) $f(\Omega)$ is a subset of the real line.
(b) $f(\Omega)$ is a subset of some line.
(c) $f(\Omega)$ is a subset of some circle.

Solution: Let $f = u + iv$.
(a) We have $v = 0$, so by the CR equations, $u_x = 0$ and $u_y = 0$ on Ω. Hence u is constant and so f is constant.
(b) Choose complex c and and angle θ so that the map $\phi(z) = \exp(i\theta)[z - c]$ sends the line to the real axis. Then $\phi(f(z))$ is real valued and analytic. So it is a constant by part (a), say $\phi(f(z)) = a$. Solving this equation for $f(z)$ then shows $f(z)$ is a constant.
(c) By composing f with a translation we can reduce to the case that the circle is centered at the origin, say $x^2 + y^2 = R^2$. So $u^2 + v^2 = R^2$. Taking partials we get

$$2uu_x + 2vv_x = 0, \quad 2uu_y + 2vv_y = 0$$

Using CR,

$$2uu_x + 2vv_x = 0, \quad -2uv_x + 2vu_x = 0$$

or

$$\begin{pmatrix} u & v \\ v & -u \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix}$$

At all points in Ω, the determinant of the matrix is $-u^2 - v^2 = -R^2$, and so is never zero. So $u_x = v_x = 0$. By CR, $u_y = v_y = 0$, and so f is constant.

2. Let Ω be an open set and f analytic on Ω. Define $\overline{\Omega} = \{\overline{z} : z \in \Omega\}$. Define

$$g(z) = \overline{f(\overline{z})}$$

Prove that g is analytic on $\overline{\Omega}$.

Solution: There are several ways to do this, e.g., power series or Cauchy Riemman equations. Here is a direct proof that just uses the definition of analytic.

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\overline{z})}}{h} = \lim_{h\to 0} \overline{\left[\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}} \right]}$$

Let $k = \overline{h}$ and this becomes

$$\overline{\lim_{k\to 0} \frac{f(\overline{z}+k) - f(\overline{z})}{k}} = \overline{f'(\overline{z})}$$

3. Let γ be the square with corners at $1+i, -1+i, -1-i, 1-i$ traversed in the counterclockwise direction. Compute

$$\int_\gamma \bar{z}\, dz$$

Solution: Modulo a few arithmetic errors, everybody did this correctly. So I won't write out the solution but just note that for each side you should get $2i$ and hence a total of $8i$.

4. Let γ be a curve with bounded variation which is not necessarily piecewise smooth. Let f be continuous on γ. Let γ^- be γ with the opposite orientation, i.e., traversed in the opposite direction. Prove that

$$\int_{\gamma^-} f(z)\, dz = -\int_\gamma f(z)\, dz$$

Note that since γ is not assumed to be smooth, you will have to use the definition of the integral I gave in class.

Solution: Some of the solutions to this one were a little more involved than they needed to be. The proposition I stated (but did not prove) in class says that as the mesh of the partition goes to zero, the "Riemann sum" approximation converges to the integral. The proposition allows non-uniform partitions but you do not need to use them in this problem. Also, we can make a specific choice for the $\tau_j = t_j$.

We can assume that the time interval for the parameterization of γ is $[0,1]$. Let P_n be the uniform partition $t_0 = 0, t_1 = 1/n, t_2 = 2/n, \cdots, t_n = 1$. Then taking $\tau_j = t_j$,

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \lim_{n\to\infty} \sum_{j=1}^{n} f(\gamma(t_j))[\gamma(t_j) - \gamma(t_{j-1})] \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} f(\gamma(j/n))[\gamma(j/n) - \gamma((j-1)/n)]
\end{aligned}$$

Now define $\gamma^-(t) = \gamma(1-t)$. Then $\gamma^-(t)$ is a parametrization of γ^-. Now we take the $\tau_j = t_{j-1}$ and the proposition says

$$\begin{aligned}
\int_{\gamma^-} f(z)\,dz &= \lim_{n\to\infty} \sum_{j=1}^{n} f(\gamma^-(t_{j-1}))[\gamma^-(t_j) - \gamma^-(t_{j-1})] \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} f(1-(j-1)/n))[\gamma(1-j/n) - \gamma(1-(j-1)/n)] \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} f((n-j+1)/n))[\gamma((n-j)/n) - \gamma((n-j+1)/n)]
\end{aligned}$$

Let $k = n - j + 1$ and this becomes

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(k/n))[\gamma((k-1)/n) - \gamma(k/n)] = -\int_{\gamma} f(z)\,dz$$

5. Let f be analytic on $\mathbb{C} \setminus \{0\}$ and suppose that

$$\int_R f(z)dz = 0$$

for all rectangles R that do not pass through 0. (R can enclose 0.) Prove that f has a primitive on $\mathbb{C} \setminus \{0\}$. You should only use results up to p. 41 in the book.

Solution: Let $\Omega_+ = \mathbb{C} \setminus (-\infty, 0]$ denote the complex plane with the non-positive real axis removed and let $\Omega_- = \mathbb{C} \setminus [0, \infty)$. For $z \in \Omega_+$, let γ_+^z be the contour from $1+i$ to z that consists of two line segments, the first being vertical and the second being horizontal. (Note that since $z \in \Omega_+$, this contour does not go through 0.) Define

$$F_+(z) = \int_{\gamma_+^z} f(w)\,dw$$

By the same argument we used in class, $F_+' = f$ on Ω_+.

Now for $z \in \Omega_-$, let γ_-^z be the contour from $-1+i$ to z that consists of two line segments, the first being vertical and the second being horizontal. Again note that the contour does not go through 0. By the same argument we used in class, $F_-' = f$ on Ω_-.

Let

$$C = \int_{[-1+i,1+i]} f(w)dw$$

where the contour is just the horizontal line segment from $-1+i$ to $1+i$. Now for $z \in \Omega_- \cap \Omega_+$,

$$\int_{[-1+i,1+i]} f(w)dw + \int_{\gamma_+} f(w)dw - \int_{\gamma_-} f(w)dw = \int_R f(w)dw = 0$$

where R is a rectangle with vertices at $-1+i$, $1+i$, $-1+iIm(z)$ and $1+iIm(z)$. (Note that $Im(z)$ is not 0 for such z.) Thus

$$C + F_+(z) = F_-(z)$$

So we can define $F(z)$ to be $F_-(z)$ on Ω_- and to be $C + F_-(z)$ on Ω_+ and then $F' = f$ on the punctured plane.
6. Book number 7 on p. 26 on "Blaschke factors."
Solution: Eveybody did this one, so I won't write out the solution.
7. Suppose a_n is a sequence of complex numbers for which $\rho = \limsup_{n\to\infty} |a_n|^{1/n}$ is not 0 or infinity. Consider

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^{-n}$$

(a) Prove that this series converges absolutely on $\{z : |z| > \rho\}$ and $f(z)$ is analytic on this set.
(b) Define

$$f_N(z) = \sum_{n=0}^{N} a_n z^{-n}$$

Prove that f_N converges uniformly to f on $\{z : |z| = r\}$ for $r > \rho$.
(c) Use (b) to prove that

$$\int_\gamma f(z)dz = \lim_{N\to\infty} \int_\gamma f_N(z)dz = \sum_{n=0}^{\infty} a_n \int_\gamma z^{-n}dz = 2\pi i a_1$$

where γ is a circle of radius r centered at the origin. Can you generalize this to other contours?

Solution: The shortest way to do this is to use what we already know about power series. You can also mimic the proofs for power series to prove the problem directly.

(a) Define

$$g(z) = \sum_{n=0}^{\infty} a_n \, z^n$$

The radius of converges of this series is $1/\rho$. So $g(z)$ is analytic on the disc of radius $1/\rho$. Since $z \to 1/z$ is analytic on the plane with the origin removed, $g(1/z)$ is analytic on $\{z : |z| > \rho\}$. We also know the power series for $g(z)$ converges absolutely and so the series for $f(z)$ does too.

(b) Define

$$g_N(z) = \sum_{n=0}^{N} a_n z^n$$

For $r > \rho$, we have $1/r < 1/\rho$ and we know that g_N converges uniformly to g on $\{z : |z| \leq 1/r\}$. Hence f_N converges uniformly to f on $\{z : |z| \geq r\}$.

(c) We know f_N converges uniformly to f on γ. Hence

$$\int_\gamma f(z) dz = \lim_{N \to \infty} \int_\gamma f_N(z) dz$$

By linearity of the integral

$$\int_\gamma f_N(z) dz = \sum_{n=0}^{N} a_n \int_\gamma z^{-n} dz = 2\pi i a_1$$

where the last equality is a computation we did in class. (You can just parametrize the circle and do the integral explicitly.)