Math 520a - Homework 2

1. Use Cauchy's integral formula (for an analytic function or its derivatives) to evaluate

(a) For the contour $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, the integral

$$\int_\gamma \frac{e^{iz}}{z^2}\, dz$$

(b) For the contour $\gamma(t) = 1 + \frac{1}{2}e^{it}$, $0 \le t \le 2\pi$, the integral

$$\int_\gamma \frac{ln(z)}{(z-1)^n}\, dz$$

2. Let $f(z)$ be an entire function such that there are constants C, D with

$$|f(z)| \le C + D|z|^n, \quad \forall z$$

Prove that f is a polynomial of degree at most n.

3. Let Ω be a region (connected open set). Suppose that f and g are analytic functions on Ω such that $f(z)g(z) = 0$ for all $z \in \Omega$. Prove that at least one of f and g is identically zero on Ω.

4. Let f be entire and suppose that for every z_0, the power series expansion about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n$$

has at least one coefficient a_n which is zero. (Note that the a_n depend on z_0.) Prove that f is a polynomial. This is problem 13 on p. 67 in the book. You can find a hint there.

5. Let D be the open disc centered at the origin with radius 1. Suppose that f is continuous on $\overline{D}$, analytic on D and that f never vanishes on $\overline{D}$. Suppose also that $|z| = 1 \Rightarrow |f(z)| = 1$. Prove that f is constant. This is problem 15 on p. 67 in the book. You can find a hint there.

6. Let $g(t)$ be continuous on $[0, \infty)$ with $\int_0^\infty |g(t)|\, dt < \infty$. Define

$$f(z) = \int_0^\infty \cos(z + t)\, g(t)\, dt$$

Prove that $f(z)$ is entire. For complex z, $\cos(z)$ is defined to be $(e^{iz}+e^{-iz})/2$. (Caution: for complex z we do not have $|\cos(z)| \leq 1$.)

7. Let Ω be open. Let f_n, f be analytic on Ω and suppose that for all circles C such that the circle and its interior are in Ω, f_n converges uniformly to f on C. Prove that f_n converges uniformly to f on all compact subsets of Ω.