Math 520a - Homework 2

1. Use Cauchy's integral formula (for an analytic function or its derivatives) to evaluate

(a) For the contour $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$, the integral

$$\int_\gamma \frac{e^{iz}}{z^2}\, dz$$

(b) For the contour $\gamma(t) = 1 + \frac{1}{2}e^{it}$, $0 \leq t \leq 2\pi$, the integral

$$\int_\gamma \frac{ln(z)}{(z-1)^n}\, dz$$

Solution: I'll just give answers for this one.

(a) -2π

(b) Integral is 0 for $n = 1$. For $n > 1$ it is $(-1)^n 2\pi i/(n-1)$.

2. Let $f(z)$ be an entire function such that there are constants C, D with

$$|f(z)| \leq C + D|z|^n, \quad \forall z$$

Prove that f is a polynomial of degree at most n.

Solution: Since f is entire it has a power series about the origin which converges for all z.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

The coefficients are given by $a_k = f^{(k)}(0)/k!$. We will show that $f^{(k)}(0) = 0$ for $k > n$. This implies $a_k = 0$ for $k > n$ and so the power series is just a polynomial.

By considering a circle of radius R, Cauchy's inequality says

$$|f^{(k)}(0)| \leq \frac{k! M_R}{R^k}$$

where M_R is the sup of $|f(z)|$ over the circle of radius R. By the hypothesis, $M_R \leq C + DR^n$. For $k > n$, $(C + DR^n)/R^k \to 0$ as $R \to \infty$, and so $f^{(k)}(0) = 0$

3. Let Ω be a region (connected open set). Suppose that f and g are analytic functions on Ω such that $f(z)g(z) = 0$ for all $z \in \Omega$. Prove that at least one of f and g is identically zero on Ω.

Solution: We can find a point $z_0 \in \Omega$ and a sequence $z_n \in \Omega$ which converges to z_0 but never equals z_0. For every n, $f(z_n)g(z_n) = 0$, and so either $f(z_n) = 0$ or $g(z_n) = 0$. So one of the sets $\{n : f(z_n) = 0\}$ and $\{n : g(z_n) = 0\}$ must be infinite. Assume the first one is infinite. Then there is a subsequence z_{n_k} with $f(z_{n_k}) = 0$. But this implies f is identically 0 on Ω.

4. Let f be entire and suppose that for every z_0, the power series expansion about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has at least one coefficient a_n which is zero. (Note that the a_n depend on z_0.) Prove that f is a polynomial. This is problem 13 on p. 67 in the book. You can find a hint there.

Solution: Note that for the power series about z_0, $a_n = 0$ is equivalent to $f^{(n)}(z_0) = 0$. So the hypothesis says that for every z_0 there is an n for which $f^{(n)}(z_0) = 0$. Now let $A_n = \{z : |z| \leq 1, f^{(n)}(z_0) = 0\}$. The union of the A_n is the unit disc and so is uncountable. So at least one A_n is uncountable (and hence infinite). Let m be such that A_m is infinite. Then there is a sequence z_l of distinct elements in A_m. Since the closed unit disc is compact, it has a convergent subsequence. Since $f^{(m)}$ vanishes on this subsequence, $f^{(m)}$ is identically zero. So f is a polynomial.

5. Let D be an open disc. Suppose that f is continuous on $\overline{D}$, analytic on D and that f never vanishes on $\overline{D}$. Suppose also that $|z| = 1 \Rightarrow |f(z)| = 1$. Prove that f is constant. This is problem 15 on p. 67 in the book. You can find a hint there.

Solution: Discussed in class.

6. Let $g(t)$ be continuous on $[0, \infty)$ with $\int_0^\infty |g(t)|\, dt < \infty$. Define

$$f(z) = \int_0^\infty \cos(z + t)\, g(t)\, dt$$

Prove that $f(z)$ is entire. For complex z, $\cos(z)$ is defined to be $(e^{iz} + e^{-iz})/2$. (Caution: for complex z we do not have $|\cos(z)| \leq 1$.)

Solution: Define

$$f_n(z) = \int_0^n \cos(z+t)\,g(t)\,dt$$

For a fixed t, $z \to \cos(z+t)\,g(t)$ is entire. Also, $\cos(z+t)\,g(t)$ is jointly continuous in t and z. By the theorem proved in class, f_n is entire. We will prove it converges uniformly on compact subsets of the plane to f. This will prove f is analytic.

Every compact subset of the plane is contained in the strip $|Im(z)| \le M$ for some $M > 0$. So it suffices to prove uniform convergence on such a strip.

$$|f(z) - f_n(z)| = |\int_n^\infty \cos(z+t)\,g(t)\,dt|$$

For $z = z + iy$, on the strip we have

$$|\cos(z+t)| = \frac{1}{2}|e^{iz+it} + e^{-iz-it}| \le \frac{1}{2}(|e^{iz}| + |e^{-iz}|) = \frac{1}{2}(|e^{-y}| + |e^{y}|) \le e^M$$

Hence

$$|f(z) - f_n(z)| \le e^M \int_n^\infty |g(t)|\,dt$$

This bound holds for all z in the strip and the right side is independent of z and goes to 0 as $n \to \infty$ proving the needed uniform convergence.

7. Let Ω be open. Let f_n, f be analytic on Ω and suppose that for all circles C such that the circle and its interior are in Ω, f_n converges uniformly to f on C. Prove that f_n converges uniformly to f on all compact subsets of Ω.

Solution: Let K be a compact subset of Ω. For each $z \in K$ we can find $\epsilon_z > 0$ such that $B_{3\epsilon_z}(z) \subset \Omega$. (Note the factor of 3.) The discs $B_{\epsilon_z}(z)$ as z ranges over K are an open cover of K. (Note there is not a factor of 3 here.) So there is a finite subcover, i.e., there are $z_1, \cdots, z_n \in K$ such that

$$K \subset \cup_{j=1}^n B_{\epsilon_{z_j}}(z_j)$$

Since there are finite number of discs in the cover, it suffices to show the convergence is uniform on each disc. To simplify the notation, let $B_\epsilon(\zeta)$ be one of the discs.

We know $B_{3\epsilon}(\zeta) \subset \Omega$. Let C be the circle centered at ζ with radius 2ϵ. Then for $z \in B_{2\epsilon}(\zeta)$, we have

$$f(z) - f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(w) - f_n(w)}{w - z} \, dw$$

If $z \in B_\epsilon(\zeta)$, we have $|w - z| \geq \epsilon$ for $w \in C$ and so

$$|f(z) - f_n(z)| \leq \frac{1}{2\pi} \frac{1}{\epsilon} |C| \, ||f - f_n||_C$$

where $||f - f_n||_C$ is the sup of $|f(w) - f_n(w)|$ over $w \in C$ and $|C|$ is the length of C which is just $4\pi\epsilon$. So

$$|f(z) - f_n(z)| \leq 2||f - f_n||_C$$

Note that the right side is now independent of z and goes to zero as $n \to \infty$ since f_n converges uniformly to f on C. This completes the proof.