Math 520a - Homework 3

1. For each of the following four functions find all the singularities and for each singularity identify its nature (removable, pole, essential). For poles find the order and principal part.

$$z\cos(z^{-1}), \quad z^{-2}\log(z+1), \quad z^{-1}(\cos(z)-1), \quad \frac{\cos(z)}{\sin(z)(e^z-1)}$$

2. In class we showed that the gamma function $\Gamma(z)$ can be analytically continued to the complex plane minus the points $0, -1, -2, -3, \cdots$. Show that this function has simple poles at $0, -1, -2, -3, \cdots$ and the residue of the pole at $-n$ is $(-1)^n/n!$.

3. Problem 7 on p. 104 of the book.

4. (a) If $f(z)$ has an isolated singularity at z_0, prove that $\exp(f(z))$ cannot have a pole there.
(b) Use (a) to show that if f has an isolated singularity at z_0 and for some positive constant c,

$$Ref(z) \le -c\log(|z-z_0|)$$

in a deleted neighborhood of z_0 then the singularity in f is removable.

5. This is problem 14 on p. 105 in the book and you can find a hint there. Prove that if $f(z)$ is entire and injective (one to one), then there are complex constants $a \neq 0, b$ such that $f(z) = az + b$.

6. In our proof of Runge's theorem we used the following proposition: Fix a compact subset of the complex plane. Let $\mathcal{A}$ be a collection of continuous functions on K such that if $f, g \in \mathcal{A}$ and $c \in \mathbb{C}$, then $cf, fg, f+g \in \mathcal{A}$. Suppose that a continuous function f can be uniformly approximated on K by functions in $\mathcal{A}$. Then any polynomial in f can be uniformly approximated on K by functions in $\mathcal{A}$.

7. Fix $w = re^{i\theta}$ with $w \neq 0$. Let γ be a curve in $\mathbb{C} \setminus \{0\}$ from 1 to w. Show that there is an integer k such that

$$\int_\gamma \frac{dz}{z} = \log(r) + i\theta + 2\pi i k$$