Math 520a - Homework 3

1. For each of the following four functions find all the singularities and for each singularity identify its nature (removable, pole, essential). For poles find the order and principal part.

Solution: $z\cos(z^{-1})$: The only singularity is at 0. Using the power series expansion of $\cos(z)$, you get the Laurent series of $\cos(z^{-1})$ about 0. It is an essential singularty. So $z\cos(z^{-1})$ has an essential singularity at 0.

$z^{-2}\log(z+1)$: The only singularity in the plane with $(-\infty,-1]$ removed is at 0. We have

$$\log(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3}\cdots$$

So

$$z^{-2}\log(z+1) = z^{-1} - \frac{1}{2} + \frac{z}{3}\cdots$$

So at 0 there is a simple pole with principal part $1/z$.

$z^{-1}(\cos(z)-1)$ The only singularity is at 0. The power series expansion of $\cos(z)-1$ about 0 is $z^2/2 - z^4/4!\cdots$, and so the singularity is removable.

$\frac{\cos(z)}{\sin(z)(e^z-1)}$ The singularities are at the zeroes of $\sin(z)$ and of e^z-1, i.e., at πn and $i2\pi n$ for integral n. These zeroes are all simple, so for $n \neq 0$ we get simple poles and at $z=0$ we get a pole of order 2. For $n \neq 0$, the residue of the simple pole at πn is

$$\lim_{z\to\pi n}(z-\pi n)\frac{\cos(z)}{\sin(z)(e^z-1)} = \frac{\cos(\pi n)}{\cos(\pi n)(e^{n\pi}-1)} = \frac{1}{e^{n\pi}-1}$$

For $n \neq 0$, the residue of the simple pole at $2\pi ni$ is

$$\lim_{z\to 2\pi ni}(z-2\pi ni)\frac{\cos(z)}{\sin(z)(e^z-1)} = \frac{\cos(2\pi ni)}{\sin(2\pi ni)} = -i\coth(2\pi n)$$

For the pole of order 2 at $z=0$ you can get the principal part by plugging in power series for the various functions and doing enough of the division to get the z^{-2} and z^{-1} terms. The principal part is $z^{-2} - \frac{1}{2}z^{-1}$.

2. In class we showed that the gamma function $\Gamma(z)$ can be analytically continued to the complex plane minus the points $0, -1, -2, -3, \cdots$. Show that this function has simple poles at $0, -1, -2, -3, \cdots$ and the residue of the pole at $-n$ is $(-1)^n/n!$.

Solution: What we did in class showed that $\Gamma(z)$ can be analytically continued to the complex plane minus the points $0, -1, -2, -3, \cdots$, and it satisfies $\Gamma(z) = \Gamma(z+1)/z$ on this domain. By a straightforward induction agrument it satisfies

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)}$$

on this domain. Now consider this equation for z in $U = \{z : 0 < |z+n| < 1\}$, a deleted neighborhood of $-n$. On U, $Re(z+n+1) > 0$, so $\Gamma(z+n+1)$ is analytic on U. Clearly $1/(z+k)$ is analytic on U. So the above equation says $\Gamma(z) = g(z)/(z+n)$ where $g(z)$ is analytic on U. Thus the pole at $-n$ is simple and the residue is $g(-n)$. Since $\Gamma(1) = 1$, the residue is

$$g(-n) = \frac{1}{(-n)(-n+1)(-n+2)\cdots(-n+n-1)} = \frac{(-1)^n}{n!}$$

3. Problem 7 on p. 104 of the book.

Solution: I just give the highlights of the computation. Let $z = e^{i\theta}$. So $dz = ie^{i\theta}d\theta$, i.e., $d\theta = -idz/z$. Then the given integral becomes

$$-4i \int_\gamma \frac{zdz}{(z^2+2az+1)^2}$$

where γ is the unit circle. There are poles at $-a \pm \sqrt{a^2-1}$. Since $a > 1$, only the pole at $-a + \sqrt{a^2-1}$ is inside circle. The zero in the denominator is of order 2, so the pole is second order. So its residue is given by

$$\begin{aligned} Res(-a+\sqrt{a^2-1}) &= \frac{d}{dz}\left[\frac{(z+a-\sqrt{a^2-1})^2 z}{(z^2+2az+1)^2}\right]_{z=a-\sqrt{a^2-1}} \\ &= \frac{d}{dz}\left[\frac{z}{(z+a+\sqrt{a^2-1})^2}\right]_{z=a-\sqrt{a^2-1}} = \frac{a}{4(a^2-1)^{3/2}} \end{aligned}$$

4. (a) If $f(z)$ has an isolated singularity at z_0, prove that $\exp(f(z))$ cannot have a pole there.

Solution: If $f(z)$ has an isolated singularity at z_0, then clearly $\exp(f(z))$ does too.

If $f(z)$ has an essential singularity, consider any non-zero $c \in \mathbb{C}$. By the Casorati-Weierstrass theorem, there is a sequence $z_n \to z_0$ such that $f(z_n) \to \log(c)$. So $\exp(f(z_n)) \to c$. Since this is true for all non-zero c, $\exp(f(z))$ must have an essential singularity at z_0.

Finally we will show that if f has a pole at z_0, then $\exp(f(z))$ has an essential singularity there. Let n be the order of the pole of f. Then

$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

where $g(z)$ is analytic and non-zero on a deleted neighborhood of z_0. Let $g(z_0) = re^{i\theta}$. Consider the sequence

$$z_k = z_0 + \frac{\exp(i\theta/n)}{n}$$

Then $f(z_k) = g(z_k)\exp(-i\theta)n^n$. Since $g(z_k) \to re^{i\theta}$, $\exp(f(z_k))$ converges to ∞. So the singularity for $\exp(f)$ is not removable. If we consider

$$w_k = z_0 + \frac{\exp(i(\pi + \theta)/n)}{n}$$

then we see that $\exp(f(z_k))$ converges to zero. Thus the singularity for $\exp(f)$ is essential.

(b) Use (a) to show that if f has an isolated singularity at z_0 and for some positive constant c,

$$Re f(z) \leq -c\log(|z - z_0|)$$

in a deleted neighborhood of z_0 then the singularity in f is removable.

Solution: The bound on f implies

$$|\exp(f(z))| = \exp(Re(f(z)) \leq \exp(-c\log(|z - z_0|)) = |z - z_0|^{-c}$$

So for large enough integers n,

$$\lim_{z \to z_0} (z - z_0)^n \exp(f(z)) = 0$$

This implies that the singularity of $\exp(f(z))$ is either a pole or removable. By (a) it cannot be a pole, so it is removable. From part (a) we know that $\exp(f(z))$ can have a removable singularity only if $f(z)$ has a removable singularity.

5. This is problem 14 on p. 105 in the book and you can find a hint there. Prove that if $f(z)$ is entire and injective (one to one), then there are complex constants $a \neq 0, b$ such that $f(z) = az + b$.

Solution: Since f is entire it has a power series about the origin that converges on the entire complex plane. First suppose this power series is a polynomial $P(z)$. If $P(z)$ has two or more distince roots, then it is not injective since is sends two different numbers to 0. So it must have a single root of order n where n is the degree of P. So $P(z) = c(z - z_0)^n$. But then $z_0 + \exp(ik2\pi/n)$ for $k = 0, 1, \cdots, n-1$ all get mapped to the same image. So n can only be 1. So $P(z)$ is linear $az + b$ and clearly a cannot be zero.

Now suppose f is not a polynomial. Then $g(z) = f(1/z)$ has an essential singularity at 0. By Casorati-Weierstrass theorem g maps $\{z : 0 < |z| < r\}$ to a dense subset of $\mathbb{C}$ for all $r > 0$. Now pick a point w_0 such that that $f(w_0) \neq 0$. Then take $r = |f(w_0)|/2$. By the density, we can find a sequence z_n with $0 < |z_n| < 1$ such that $g(z_n)$ converges to $f(w_0)$. Since the sequence z_n is bounded, it has a convergent subsequence z_{n_k} converges to some w. Note that $|w| \leq r/2$. By continuity $g(w) = f(w_0)$. So $f(1/w) = f(w_0)$. Since f is injective, $1/w = w_0$. But $|w_0| = r$ and $|w| \leq r/2$, a contradiction.

6. In our proof of Runge's theorem we used the following proposition: Fix a compact subset of the complex plane. Let $\mathcal{A}$ be a collection of continuous functions on K such that if $f, g \in \mathcal{A}$ and $c \in \mathbb{C}$, then $cf, fg, f + g \in \mathcal{A}$. Suppose that a continuous function f can be uniformly approximated on K by functions in $\mathcal{A}$. Then any polynomial in f can be uniformly approximated on K by functions in $\mathcal{A}$.

Solution: Let $P(z)$ be a polynomial. We want to show that $P(f(z))$ can be uniformly approximated on K by functions in $\mathcal{A}$. Let $\epsilon > 0$.

Since f is continuous and K is compact, $f(K)$ is compact. So $P(z)$ is uniformly continuous on $f(K)$. So there is a $\delta > 0$ so that $z, w \in f(K)$ and $|z - w| < \delta$ implies $|P(z) - P(w)| < \epsilon$. Let $g \in \mathcal{A}$ be such that $|f(z) - g(z)| < \delta$ for $z \in K$. Then since $f(z), g(z) \in f(K)$, this implies

$|P(f(z)) - P(g(z))| < \epsilon$. The properties on $\mathcal{A}$ imply that $P(g(z)) \in \mathcal{A}$. So this completes the proof.

7. Fix $w = re^{i\theta}$ with $w \neq 0$. Let γ be a curve in $\mathbb{C} \setminus \{0\}$ from 1 to w. Show that there is an integer k such that

$$\int_\gamma \frac{dz}{z} = \log(r) + i\theta + 2\pi ik$$

Solution: I assume $r \geq 1$. The changes for the other case of $r < 1$ are minor. Let γ_1 be the contour that is the line segment from 1 to r: $\gamma_1(t) = t, 1 \leq t \leq r$. Then $\int_{\gamma_1} dz/z$ is $\log(r)$.

Let γ_2 be the contour that is the subarc of the circle of radius r from r to $re^{i\theta}$. So $\gamma_2(t) = re^{it}, 0 \leq t \leq \theta$. Then

$$\int_{\gamma_2} \frac{dz}{z} = \int_0^\theta \frac{\gamma_2'(t)}{\gamma_2(t)} dt = \int_0^\theta i dt = i\theta$$

Let $\gamma - \gamma_2 - \gamma_1$ be the contour that follows γ from 1 to w, then follows γ_2 backwards from w to r and then follows γ_1 backwards from r to 1. This is a closed contour, so the integral of $1/z$ around this contour is $2\pi ik$, where k is the winding number of this contour. So

$$\int_\gamma \frac{dz}{z} = 2\pi ik + \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} = 2\pi ik + \log(r) + i\theta$$