Math 520a - Homework 4

1. Problem 5 on page 103 in the book.

2. Problem 12 on page 105 in the book.

3. Suppose that f is analytic on some annulus centered at 0. So it has a Laurent series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Let

$$R_1 = \inf\{r_1 : for\, some\, r_2 > 0,\ f\, is\, analytic\, on\, r_1 < |z| < r_2\}$$

$$R_2 = \sup\{r_2 : for\, some\, r_1 > 0,\ f\, is\, analytic\, on\, r_1 < |z| < r_2\}$$

Prove that

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{1/n}$$

$$\frac{1}{R_2} = \limsup_{n\to\infty} |a_n|^{1/n}$$

4. Let f and g be analytic on an open set containing the closed disc $|z| \leq 1$. Suppose f has a simple zero at $z = 0$ and has no other zeroes in the closed disc. Define for complex w,

$$f_w(z) = f(z) + wg(z)$$

Prove that there is an $\epsilon > 0$ such that for $|w| < \epsilon$, f_w has a unique zero z_w in the closed disc and the mapping $w \to z_w$ is continuous.

5. Let f be analytic on the complex plane except for isolated singularites at $z_1, z_2, \cdots, z_m$. Define the residue of f at ∞ to be the residue of $-z^{-2}f(1/z)$ at $z = 0$. Let $R = \max_j |z_j|$.

(a) Express the residue at ∞ in terms of the coefficients of the Laurent series of f in the region $\{z : R < |z|\}$.
(b) What is the relation of the residue at ∞ to the integral

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz$$

where $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$ for $r > R$?
(c) Show that

$$Res(f, \infty) = -\sum_{k=1}^{m} Res(f, z_k)$$

6. Let

$$f(z) = \frac{\cos z}{z(1+z^2)}$$

(a) Find the z^5 term in the Laurent series of f in the annulus $\{z : 0 < |z| < 1\}$.
(b) Find the z^{-5} term in the Laurent series of f in the annulus $\{z : 1 < |z| < \infty\}$.

7. Let $\overline{D} = \{z : |z| \leq R\}$. Let f and g be analytic on an open set containing $\overline{D}$. Suppose that $|f(z)| = |g(z)|$ when $|z| = R$, and that f and g never vanish on $\overline{D}$. Prove that there is a constant c with $|c| = 1$ such that $f(z) = cg(z)$.