Math 520a - Homework 4 - Selected solutions

1. Problem 5 on page 103 in the book.

Solution: I'll just make a comment on this one. You close the contour with a semicircle in either the upper or lower half plane depending on the sign of ϵ. Most people worked out both cases. But note that the change of variables $x \to -x$ in the original integral shows that the integral is equal to the integral with ϵ replaced by $-\epsilon$. In other words the integral is an even function of ϵ. So you only have to compute it for one case, e.g., $\epsilon \geq 0$.

2. Problem 12 on page 105 in the book.

Solution: Inside the given circle there is a second order pole at $-u$ and first order poles at k for $|k| \leq N$. The residue from the pole at $-u$ leads to the $\pi^2/\sin^2(\pi u)$ and the sum of the other residues leads to the partial sum

$$\sum_{n=-N}^{N} \frac{1}{(u+n)^2}$$

This part was pretty straightforward and I won't write it out. The harder part is showing that the integral around the contour converges to zero as $N \to \infty$.

We will show that there is a constant M (independent of N) such that $|\cot(\pi z)| \leq M$ for all N and $|z| = N + 1/2$.

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i\frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1}$$

Let $z = x + iy$. Then this becomes

$$i\frac{e^{2i\pi x - 2\pi y} + 1}{e^{2i\pi x - 2\pi y} - 1}$$

Since $\cot(\pi z)$ is an odd function, we need only consider z in the upper half plane, ie., $y \geq 0$. If x is within $1/4$ of $N + 1/2$ for some integer N, then $\cos(2\pi x) \leq 0$. So the real part of $e^{2i\pi x - 2\pi y}$ is ≤ 0. Hence $|e^{2i\pi x - 2\pi y} - 1| \geq 1$. The numerator is trivially bounded in modulus by 2. So $|\cot(\pi z)| \leq 2$ on the vertical strips given by $|Re(z) - (N+1/2)| < 1/4$ for some N. The parts of the circle that do not lie in these strips are bounded away from the real axis, i.e., there is a $\delta > 0$ such that they lie in $\{z : |Im(z)| \geq \delta\}$. For $y \geq \delta$,

$$|i\frac{e^{2i\pi x-2\pi y}+1}{e^{2i\pi x-2\pi y}-1}| \leq \frac{e^{-2\pi y}+1}{1-e^{-2\pi y}} \leq \frac{e^{-2\pi\delta}+1}{1-e^{-2\pi\delta}}$$

With this bound on $|\cot(\pi z)|$, showing the integral around the circle of radius $N+1/2$ goes to zero as $N \to \infty$ is straightforward.

3. Suppose that f is analytic on some annulus centered at 0. So it has a Laurent series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Let

$$R_1 = \inf\{r_1 : for\, some\, r_2 > 0,\ f\, is\, analytic\, on\, r_1 < |z| < r_2\}$$

$$R_2 = \sup\{r_2 : for\, some\, r_1 > 0,\ f\, is\, analytic\, on\, r_1 < |z| < r_2\}$$

Prove that

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{1/n}$$

$$\frac{1}{R_2} = \limsup_{n\to\infty} |a_n|^{1/n}$$

Solution: This problem was so poorly stated that everyone missed the point of I was trying to get at. So I gave everyone full credit on the problem and wrote comments on most papers. I will reassign this problem (hopefully better stated) in the problem set after the midterm. Here is an attempt at a better statement of the problem:

Suppose that f is analytic on the annulus $\{z : \rho_1 < |z| < \rho_2\}$. From what we did in class we know that it has a Laurent series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

meaning that the series converges to $f(z)$ on the annulus. Moreover the convergence is absolute.

Define

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{1/n}$$

$$\frac{1}{R_2} = \limsup_{n\to\infty} |a_n|^{1/n}$$

(a) Prove that $R_1 \le \rho_1$ and that $R_2 \ge \rho_2$.
(b) Prove that the Laurent series converges absolutely on $\{z : R_1 < |z| < R_2\}$ and uniformly on compact subsets of this set, and so defines an analytic continuation of f to this annulus.
(c) Prove that if f has an analytic continuation to an annulus $\{z : r_1 < |z| < r_2\}$ with $r_1 \le R_1$ and $r_2 \ge R_2$, then $r_1 = R_1$ and $r_2 = R_2$. In other words the annulus in part (b) is the largest annulus (about 0) containing the original annulus on which f has an analytic continuation.

4. Let f and g be analytic on an open set containing the closed disc $|z| \le 1$. Suppose f has a simple zero at $z = 0$ and has no other zeroes in the closed disc. Define for complex w,

$$f_w(z) = f(z) + wg(z)$$

Prove that there is an $\epsilon > 0$ such that for $|w| < \epsilon$, f_w has a unique zero z_w in the closed disc and the mapping $w \to z_w$ is continuous.

Solution: Let $M = \sup |g(z)|$, $m = \inf |f(z)|$ where the sup and inf are over the unit circle. Since the circle is compact and both functions are continuous, the sup and inf are attained and so $M < \infty$ and $m > 0$. Assume $M > 0$. (Otherwise g vanishes on the unit circle and so must be the zero function.) Define $\epsilon = m/M$. Let w be such that $|w| < \epsilon$. We apply Rouché's theorem to the functions $f(z)$ and $wg(z)$. We have $|wg(z)| < \epsilon M = m$ and $|f(z)| \ge m$ on the circle, so $|wg(z)| < |f(z)|$ on the circle. By the theorem $f(z)$ and $f(z) + wg(z)$ have the same number of zeros inside the circle. So $f(z) + wg(z)$ has exactly one zero, z_w.

To prove z_w is continuous in w, fix w with $|w| < \epsilon$ and let w_n be a sequence converging to w. Suppose z_{w_n} does not converge to z_w. Then there is an $\epsilon > 0$ and a subsequence of z_{w_n} whose distance to z_w is always at least ϵ. This subsequence is in the unit disc, a bounded set, so it has a subsequence which converges to something, call it z', in the closed unit disc. Note that z' cannot be z_w. Let u_k be the corresponding subsequence of the subsequence of w_n. So z_{u_k} converges to z'. Now w_{u_k} converges to w, so $f_{w_{u_k}}(z_{u_k})$ converges to $f_w(z')$. But $f_{w_{u_k}}(z_{u_k}) = 0$ for all k, so z' is a root of f_w. By the bounds in the first paragraph, f_w is not zero on the boundary of the disc, so z' is in the open unit disc. But f_w has only one root in this disc, and it is z_w. Contradiction.

5. Let f be analytic on the complex plane except for isolated singularites at $z_1, z_2, \cdots, z_m$. Define the residue of f at ∞ to be the residue of $-z^{-2}f(1/z)$ at $z = 0$. Let $R = \max_j |z_j|$.
(a) Express the residue at ∞ in terms of the coefficients of the Laurent series of f in the region $\{z : R < |z|\}$.

Solution: Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent series of f for $|z| > R$. Then for $|z| < 1/R$,

$$f(1/z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

So

$$-z^{-2}f(1/z) = -\sum_{n=-\infty}^{\infty} a_n z^{-n-2}$$

The residue of this is the coef of the $1/z$ term. This is when $n = -1$ and the residue at ∞ is $-a_{-1}$.
(b) What is the relation of the residue at ∞ to the integral

$$\frac{1}{2\pi i} \int_\gamma f(z) dz$$

where $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$ for $r > R$?

Solution: The Laurent series converges uniformly on the circle of radius r, so we can integrate it term by term to get

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = a_{-1}$$

So this integral is minus the residue at ∞.

(c) Show that

$$Res(f,\infty) = -\sum_{k=1}^{m} Res(f, z_k)$$

Solution: The contour in part (b) encloses all the singularites with winding number 1, so by the residue theorem the integral in part (b) is $\sum_{k=1}^{m} Res(f, z_k)$, and the equation in (c) follows.

6. Let

$$f(z) = \frac{\cos z}{z(1+z^2)}$$

(a) Find the z^5 term in the Laurent series of f in the annulus $\{z : 0 < |z| < 1\}$.

Solution:

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$
$$\frac{1}{1+z^2} = 1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6}\cdots$$

So the power series of $\cos z/(1+z^2)$ about the origin is

$$(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots)(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6}\cdots)$$

The z^5 term in the Laurent series we want will be $1/z$ times the z^6 term in this power series which works out to be $(-1 - 1/2 - 1/24 - 1/720)z^5 = -\frac{1111}{720}z^5$.
(b) Find the z^{-5} term in the Laurent series of f in the annulus $\{z : 1 < |z| < \infty\}$.

Solution: If we take $r > 1$, the coefficient of z^5 in the Laurent series is given by

$$\frac{1}{2\pi i}\int_{|z|=r} z^4 f(z)dz = \frac{1}{2\pi i}\int_{|z|=r} \frac{z^3\cos(z)}{1+z^2}dz$$

You can compute this integral by the residue theorem and find $\cos(i) = \cosh(1)$.

7. Let $\overline{D} = \{z : |z| \leq R\}$. Let f and g be analytic on an open set containing $\overline{D}$. Suppose that $|f(z)| = |g(z)|$ when $|z| = R$, and that f and g never vanish on $\overline{D}$. Prove that there is a constant c with $|c| = 1$ such that $f(z) = cg(z)$ for $|z| \leq 1$.

Solution: Let $h(z) = f(z)/g(z)$. Since g does not vanish on $D = \{z : |z| < R\}$, h is analytic on D. On the boundary of D we have $|h(z)| = 1$. So by the maximum-modulus theorem we have $|h(z)| \leq 1$ on D, i.e., $|f(z)| \leq |g(z)|$ on D. The same argument with f and g interchanged shows $|g(z)| \leq |f(z)|$ on D. So $|g(z)| = |f(z)|$ on D. Thus the analytic function h attains it maximum modulus at an interior point (in fact at every interior point) and so must be a constant. So $f(z) = cg(z)$ and $|c|$ must be 1.