Math 520a - Homework 5

1. Is it possible to define a branch of the logarithm $f(z)$ such that for all positive integers n, $f(n) = \log(n) + 2\pi i n$? You should justify your answer, i.e, show it cannot be done or show how to do it.

2. Let

$$f(z) = \frac{1}{2}(z + \frac{1}{z})$$

Let $U = \{z \in \mathbb{H} : |z| > 1\}$. Show that f is a conformal map of U onto the upper half plane $\mathbb{H}$.

3. Book, page 250, problem 11.

4. Let U be a simply connected region which is not empty and not the entire plane. Let $z_0 \in U$.
(a) Prove there is a unique $r > 0$ such that there is a conformal map f from U onto the disc with radius r centered at the origin satisfying $f(z_0) = 0$, $f'(z_0) = 1$. The radius r is called the *conformal radius* of U (with respect to z_0). We will denote it by $r(U, z_0)$.
(b) Let U_1 and U_2 be simply connected regions which are not empty and not the entire plane. Suppose that $U_1 \subset U_2$ and $z_0 \in U_1$. Prove that $r(U_1, z_0) \leq r(U_2, z_0)$.

5. Suppose that f is analytic on the annulus $\{z : \rho_1 < |z| < \rho_2\}$. From what we did in class we know that it has a Laurent series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

meaning that the series converges to $f(z)$ on the annulus. Moreover the convergence is absolute.

Define

$$R_1 = \limsup_{n\to\infty} |a_{-n}|^{1/n}$$

$$\frac{1}{R_2} = \limsup_{n\to\infty} |a_n|^{1/n}$$

(a) Prove that $R_1 \leq \rho_1$ and that $R_2 \geq \rho_2$.
(b) Prove that the Laurent series converges absolutely on $\{z : R_1 < |z| < R_2\}$ and uniformly on compact subsets of this set, and so defines an analytic continuation of f to this annulus.
(c) Prove that if f has an analytic continuation to an annulus $\{z : r_1 < |z| < r_2\}$ with $r_1 \leq R_1$ and $r_2 \geq R_2$, then $r_1 = R_1$ and $r_2 = R_2$. In other words the annulus in part (b) is the largest annulus (about 0) containing the original annulus on which f has an analytic continuation.

6. Moibius transformations give homeomorphisms of the Riemann sphere. Find all Moibius transformations that corresponds to rotations of the sphere.

7. Problem 1 on page 108 of the book leads us through a proof of the famous Koebe 1/4 theorem. (The book calls it the Koebe-Bierbach theorem.) Do at least parts (a) and (b).