Math 520a - Homework 5 - Selected solutions

1. Is it possible to define a branch of the logarithm $f(z)$ such that for all positive integers n, $f(n) = \log(n) + 2\pi in$? You should justify your answer, i.e, show it cannot be done or show how to do it.

Solution: Yes, it is possible. Let ω be the spiral which in polar coordinates is $\theta = (r + 1/2)/2\pi$. It spirals counterclockwise and crosses the positive real axis at $n + 1/2$ for $n = 0, 1, 2, 3, \cdots$. Let $g(z)$ be the branch of the log that is given by integrating $1/w$ from 1 to z along any contour γ that does not cross ω. Clearly $g(1) = 0$. By drawing pictures we see that if we take a contour from 1 to n which does not cross ω and close it by going from n back to 1 along the real axis, then the closed countour has winding number $n - 1$. So

$$2\pi i(n-1) = \int_\gamma \frac{dw}{w} + \int_n^1 \frac{dw}{w} = g(n) - log(n)$$

So $g(n) = log(n) + 2\pi i(n-1)$. Now let $f(z) = g(z) + 2\pi i$ to get the branch of the log called for in the problem.

2. Let

$$f(z) = \frac{1}{2}(z + \frac{1}{z})$$

Let $U = \{z \in \mathbb{H} : |z| > 1\}$. Show that f is a conformal map of U onto the upper half plane $\mathbb{H}$.

Solution: Clearly f is analytic everywhere but 0.

$$Im(f(x+iy)) = \frac{1}{2}\left[y + \frac{-y}{x^2+y^2}\right] = \frac{y}{2}\frac{(x^2+y^2)-1}{x^2+y^2}$$

If $x + iy \in U$, then $y > 0$ and $x^2 + y^2 > 1$, so the above is positive, ie., $f(x + iy) \in \mathbb{H}$.

To show it is onto an injective, let $w \in \mathbb{H}$. Then $f(z) = w$ is a quadratic equation for z with solutions $z = w \pm \sqrt{w^2 - 1}$. Some algebra shows that for $w \in \mathbb{H}$, exactly one of these two solutions lies in U.

3. Book, page 250, problem 11.

Solution: We assume f is not constant. By the maximum value principle this implies $|f(z)| < M$ on the circle $|z| = R$. Let $g(z) = f(Rz)/M$. Then g maps $\mathbb{D}$ into $\mathbb{D}$. Let

$$\psi(z) = \frac{z - g(0)}{1 - \overline{g(0)}z}$$

As we have seen before ψ is a Moibus transformation of $\mathbb{D}$ onto itself which sends $g(0)$ to 0. So $\psi(g(z)) = \psi(f(Rz)/M)$ maps $\mathbb{D}$ into itself, sending 0 to 0. By Schwarz's lemma, $|\psi(g(z))| \leq |z|$ for all $z \in \mathbb{D}$. So using $g(0) = f(0)/M$ and the past definitions,

$$\frac{\frac{f(Rz)}{M} - \frac{f(0)}{M}}{1 - \frac{\overline{f(0)}f(Rz)}{M^2}} \leq |z|$$

Letting $Rz = w$, and simplifying we have

$$\frac{f(w) - f(0)}{M^2 - \overline{f(0)}f(w)} \leq \frac{|w|}{MR}$$

4. Let U be a simply connected region which is not empty and not the entire plane. Let $z_0 \in U$.
(a) Prove there is a unique $r > 0$ such that there is a conformal map f from U onto the disc with radius r centered at the origin satisfying $f(z_0) = 0$, $f'(z_0) = 1$. The radius r is called the *conformal radius* of U (with respect to z_0). We will denote it by $r(U, z_0)$.
(b) Let U_1 and U_2 be simply connected regions which are not empty and not the entire plane. Suppose that $U_1 \subset U_2$ and $z_0 \in U_1$. Prove that $r(U_1, z_0) \leq r(U_2, z_0)$.

Solution: (a) By the Riemann mapping theorem there is a conformal map F from U onto the unit disc with $F(z_0) = 0$ and $F'(z_0) > 0$. We will show $r = 1/F'(z_0)$. The map $f(z) = rF(z)$ maps U onto the disc of radius r and $f'(z_0) = rF'(z_0) = 1$. Suppose g is a conformal map of U onto a disc of radius r' with $g(z_0) = 0$ and $g'(z_0) = 1$. Then $G(z) = g(z)/r'$ is a conformal map of U onto the unit disc with $G(z_0) = 0$ and $G'(z_0) > 0$. By the uniqueness part of the Riemann mapping theorem, $G(z) = F(z)$. In particular, $G'(z_0) = F'(z_0)$. Since $g'(z_0) = f'(z_0) = 1$, this implies $r = r'$.

(b) Let F_i be the unique conformal maps from U_i onto $\mathbb{D}$ with $F_i(z_0) = 0$ and $F_i'(z_0) > 0$. By part (a), $r_i = 1/F'(z_0)$. So we want to show $F_1'(z_0) \geq F_2'(z_0)$. From the proof of the Riemann mapping theorem, $F_1'(z_0) \geq F'(z_0)$ for any conformal map of U_1 to a subset of $\mathbb{D}$. If we restrict F_2 to U_1, then it is such a conformal map and the bound follows.

6. Moibius transformations give homeomorphisms of the Riemann sphere. Find all Moibius transformations that corresponds to rotations of the sphere.

Solution: Rotations about the z-axis by angle θ correspond to the Moibius

transformation $f(z) = e^{i\theta} z$. We denote this rotation of the sphere by R^z_θ. The Moibius transformation corresponds to the two by two matrix

$$R^z_\theta \sim \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

(Remember that the matrix should have determinant 1.)

Let R^y_ϕ be the rotation of the sphere about the axis parallel to the y-axis. Let C be the great circle on the sphere that projects to the real axis. So $R^y_\phi(C) = C$. The points fixed by R^y_ϕ project to i and $-i$ in the complex plane. So R^y_ϕ corresponds to a Moibius transformation that maps $\mathbb{R}_\infty$ to itself and fixes $\pm i$. The first condition implies it is of the form

$$f(z) = \frac{az+b}{cz+d}$$

with a, b, c, d real. A little calculation shows that fixing i and $-i$ implies $b = -c$ and $a = d$. The determinant of the corresponding two by two matrix is $ad - bc = a^2 + b^2$. This must be 1, so we can write a and b as $a = \cos(\omega)$, $b = -\sin(\omega)$ for some ω. (The $-$ is for latter convience.) Take the image of the north pole under the rotation, project it to the plane and compare this with what the Moibius transformation does to ∞ and you find that $\omega = \phi/2$. (Note that $\phi = 2\pi$ corresponds to the identity rotation, but the two by two matrix is minus the identity. This is ok, since minus the two by two identity corresponds to the identity Moibuis transformation.) So

$$R^y_\phi \sim \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}$$

Now take a general rotation R of the sphere about the axis L. We can use a R^z to rotate L so that it hits the great circle C. Then we can use a R^y to rotate it to the line through the north and south poles. It follows that R is of the form

$$R = R^z_{-\theta} R^y_{-\phi} R^z_\alpha R^y_\phi R^z_\theta$$

for some θ, ϕ, α. Note that the two by two matrices corresponding to R^z and to R^y are unitary. It follows that the two by two matrix corresponding to R is unitary. So the Moibius transformations we get all correspond to two by two unitary matrices. Since the two by two matrices have determinant one, we have shown that the Mobius transformations map into $SU(2)$ inside $SL(2, \mathbb{C})$. To show that we get all of $SU(2)$ we need to show that the two by two matrix correspond to R above gives all of $SU(2)$. I have not checked this. Note that a general matrix in $SU(2)$ is of the form

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

Recall that the homorphism of $SL(2, \mathbb{C})$ to Moibius transformations has kernel $I, -I$. $SU(2)$ is the universal covering group of $SO(3)$, the group of rotations, and it is a two fold cover. So this all fits together nicely.