**Math 520a - Homework 6**

1. Let $\Omega$ be an open subset of the complex plane that is symmetric about the real axis and intersects the real axis. Let $\Omega^+ = \{z \in \Omega : Im(z) > 0\}$. Let $I = \{z \in \Omega : Im(z) = 0\}$, the intersection of $\Omega$ with the real axis. Let $f$ be analytic on $\Omega^+$ and continuous on $\Omega^+ \cup I$. Suppose that $|f(z)| = 1$ for $z \in I$. Prove that $f$ has an analytic continuation to $\Omega$.

2. Problem 2 on page 109 in the book.

3. Let $0 < k < 1$ and define

$$f(z) = \int_0^z \frac{dw}{[(1-w^2)(1-k^2w^2)]^{1/2}}$$

(a) The branch cut for the square root is "chosen so that the denominator is positive when $w$ is real and $-1 < w < 1$." Give an explicit definition of the branch cut that does this.
(b) Find the image of $\mathbb{H}$ under $f$ and show that $f$ is a conformal map between $\mathbb{H}$ and this region. You can use all the things we have proved about Schwarz-Christoffel maps, but pay attention to branch cuts. (This is an example in the book if you get stuck.)

4. Problem 22 on page 253 in the book. I would ignore the book's hint and instead use the corresponding result for the half plane and a Moibus transformation between $\mathbb{H}$ and $\mathbb{D}$.

5. Problem 23 on page 253 in the book.

6. Let $\Omega$ be a bounded simply connected region whose boundary is a piecewise smooth curve. Let $f$ be a continuous function on the boundary. Consider the Dirichlet problem

$$\Delta u(z) = 0, \quad z \in \Omega$$
$$u(z) = f(z), \quad z \in \partial\Omega$$

Prove that the solution is unique. Hint: mean value property.

7. Problem 8 on page 249 in the book.

8. (a) Prove that every meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.

(b) Let $a_n$ and $b_n$ be sequences of complex numbers that do not have a limit point. We assume $a_n \neq b_m$ for all $n, m$. There can be repetitions within the sequences, but a given complex number only occurs a finite number of times in each sequence. Prove there is a meromorphic function with zeroes at the $a_n$ and nowhere else and poles at the $b_n$ and nowhere else. Furthermore, the order of the zero at $a$ is the number of times $a$ appears in $a_n$ and the order of the pole at $b$ is the number of times $b$ appears in $b_n$.