Math 520a - Homework 6 - selected solutions

1. Let Ω be an open subset of the complex plane that is symmetric about the real axis and intersects the real axis. Let $\Omega^+ = \{z \in \Omega : Im(z) > 0\}$. Let $I = \{z \in \Omega : Im(z) = 0\}$, the intersection of Ω with the real axis. Let f be analytic on Ω^+ and continuous on $\Omega^+ \cup I$. Suppose that $|f(z)| = 1$ for $z \in I$. Prove that f has an analytic continuation to Ω.

Solution: The problem is not true as stated. We need to also assume f does not vanish on Ω^+. We have already seen that $\overline{f(\overline{z})}$ is analytic on the relection of Ω^+ about the real axis,which we denote by Ω^-. If f never vanishes on Ω^+, then $g(z) = 1/\overline{f(\overline{z})}$ is analytic Ω^-. Note that f is continuous on $\Omega^+ \cup I$ and g is continuous on $\Omega^- \cup I$. On I, $|f(z)| = 1$ implies $f(z) = g(z)$. By the same argument used to prove the Schwarz reflection principle, g provides the analytic continuation.

2. Problem 2 on page 109 in the book.

Solution: Define

$$\phi(z) = \frac{z_0 - z}{1 - \overline{z_0} z}$$

so ϕ maps the unit disc to the unit disc and sends z_0 to 0. Consider $v(z) = u(\phi(z))$. It is a harmonic function on the disc, so by the mean value property,

$$v(0) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\theta})\, d\theta$$

We have $v(0) = u(\phi(0)) = u(z_0)$. So

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(\phi(e^{i\theta}))\, d\theta$$

Do a change of variables in the integral given by

$$e^{i\alpha} = \frac{z_0 - e^{i\theta}}{1 - \overline{z_0} e^{i\theta}}$$

Some calculation then yields the result.

3. Let $0 < k < 1$ and define

$$f(z) = \int_0^z \frac{dw}{[(1 - w^2)(1 - k^2 w^2)]^{1/2}}$$

(a) The branch cut for the square root is “chosen so that the denominator is positive when w is real and $-1 < w < 1$.” Give an explicit definition of the branch cut that does this.
(b) Find the image of $\mathbb{H}$ under f and show that f is a conformal map between $\mathbb{H}$ and this region. You can use all the things we have proved about Schwarz-Christoffel maps, but pay attention to branch cuts. (This is an example in the book if you get stuck.)

Solution: The problem is worked out in examples in the book.

4. Problem 22 on page 253 in the book. I would ignore the book's hint and instead use the corresponding result for the half plane and a Moibus transformation between $\mathbb{H}$ and $\mathbb{D}$.

Solution: Let

$$\phi(z) = \frac{i-z}{i+z}$$

so ϕ maps $\mathbb{H}$ to $\mathbb{D}$. So $F \circ \phi$ is a conformal map of $\mathbb{H}$ to P. Thus there are real numbers $A_1, A_2, \cdots A_n$ and complex constants c_1, c_2 such that

$$F \circ \phi(z) = c_1 \int_0^z \frac{dw}{(w-A_1)^{\beta_1} \cdots (w-A_n)^{\beta_n}} + c_2$$

Do a change of variables $\zeta = \phi(w)$ and after some algebra you get the formula in the book.

5. Problem 23 on page 253 in the book.

Solution: I don't know how to do this one.

6. Let Ω be a bounded simply connected region whose boundary is a piecewise smooth curve. Let f be a continuous function on the boundary. Consider the Dirichlet problem

$$\begin{aligned} \Delta u(z) &= 0, \quad z \in \Omega \\ u(z) &= f(z), \quad z \in \partial\Omega \end{aligned}$$

Prove that the solution is unique. Hint: mean value property.

Solution: The hint should have said maximum value property not mean value property. Let u_1 and u_2 be two solutions. Then $u = u_1 - u_2$ is a harmonic function that vanishes on the boundary. By the maximum value property u attains its max on the boundary. So that max is 0. Similarly the min is zero. So $u = 0$, i.e., $u_1 = u_2$.

7. Problem 8 on page 249 in the book.

Solution:

8. (a) Prove that every meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.

Solution: Let $f(z)$ be meromorphic. Let a_n be its poles, listed according to their order. Weiestrass's theorem says there is an entire function g whose zeroes (listed according to multiplicity) are exactly the a_n. Now look at $f(z)g(z)$. It is analytic except possible at the a_n. Let a be in this list and let m be the number of times it appears in the list Then there is a neighborhood of a in which we have $f(z) = F(z)/(z-a)^m$ where F is analytic near a, and $g(z) = G(z)(z-a)^m$ where G is analytic near a. So fg has a removable singularity at a. Thus $h(z) = f(z)g(z)$ is an entire funtion. So $f = h/g$.

(b) Let a_n and b_n be sequences of complex numbers that do not have a limit point. We assume $a_n \neq b_m$ for all n, m. There can be repetitions within the sequences, but a given complex number only occurs a finite number of times in each sequence. Prove there is a meromorphic function with zeroes at the a_n and nowhere else and poles at the b_n and nowhere else. Furthermore, the order of the zero at a is the number of times a appears in a_n and the order of the pole at b is the number of times b appears in b_n.

Solution: Let $f(z)$ be an entire functions with zeros at $\{a_n\}$ where the multiplicity of the zero is the number of times the point appear in the list. Similarly, let $g(z)$ be entire with zeroes at b_n. Now consider $f(z)/g(z)$. The condition that $a_n \neq b_m$, means that at a zero of f, g is not zero. So the quotient has zeroes at the zeroes of f with the same multiplicity. Whereever g has a pole, f is not zero, so the quotient has a pole where g has a zero and the order of the pole is the order of the zero.