Math 520a - Midterm take home exam

Do 5 out of the 6 problems. Do not turn in more than 5.

The exam is due Monday, Nov 2 at the start of class. Late papers will only be accepted in the case of illness. You may consult the textbook, your class notes and other books as long as you are not looking up the actual problem. You may not consult other people or the web. You can ask me to clarify the problem statement, but I will not give hints.

1. Find all entire analytic functions satisfying

$$|f(z)| \leq \frac{|z|}{\log(|z|)}, \quad for \quad |z| > 1$$

2. Prove that if f is an entire analytic function such that

$$f(z) = f(z+1) = f(z+\sqrt{2}), \quad \forall z \in \mathbb{C}$$

then f is constant.

3. Let f be analytic on an open set U. Let $z_1, z_2, \cdots z_n$ be the distinct zeroes of f and $m_1, m_2, \cdots, m_n$ their multiplicities. Let γ be a closed contour which is homotopic in U to a point. Compute the following integral in terms of z_j, m_j and the winding numbers $n(\gamma, z_j)$, and justify your computation.

$$\int_\gamma \frac{z f'(z)}{f(z)} dz$$

4. Let f be analytic on a region U. Define

$$M = \sup_{z \in U} Ref(z)$$

Suppose there is a $z_0 \in U$ with $Ref(z_0) = M$. Prove that f is a constant.

5. Let $\log(z)$ be the branch of the log with branch cut along the negative real axis. It has a power series about $z_0 = -1 + i$:

$$log(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

A confused professor claims that since the distance from z_0 to the branch cut is 1, the radius of convergence of the power series is 1. Explain what is wrong with this reasoning and find (with justification) the correct radius of convergence.

6. Suppose that f is analytic in a disc of radius ρ about 0 and has a zero of order n at 0. Suppose that f has no other zero in this disc. Define a closed contour by $\gamma(t) = f(re^{it})$ for $0 \leq t \leq 2\pi$, with $r < \rho$. Compute the winding number of γ about 0 in terms of n and r and justify your computation.