Math 520a - Midterm take home exam

Do 5 out of the 6 problems. Do not turn in more than 5.

The exam is due Monday, Nov 2 at the start of class. Late papers will only be accepted in the case of illness. You may consult the textbook, your class notes and other books as long as you are not looking up the actual problem. You may not consult other people or the web. You can ask me to clarify the problem statement, but I will not give hints.

1. Find all entire analytic functions satisfying

$$|f(z)| \leq \frac{|z|}{\log(|z|)}, \quad for \quad |z| > 1$$

Solution: We will show that $f'(z) = 0$ for all z. By Cauchy's integral formula, for $R > |z|$,

$$f'(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^2} dw$$

Since $|w - z| \geq R - |z|$,

$$|f'(z)| \leq \frac{1}{2\pi} \frac{R}{log(R)} \frac{1}{(R-|z|)^2} 2\pi R = \frac{R^2}{log(R)(R-|z|)^2}$$

As $R \to \infty$, this goes to zero. Thus any function satifying the conditions must be a constant.

A little calculus shows that the mininum of $x/\log(x)$ over $x > 1$ is e. So the functions satisfying the conditions are $f(z) = c$ where $|c| \leq e$.

2. Prove that if f is an entire analytic function such that

$$f(z) = f(z+1) = f(z+\sqrt{2}), \quad \forall z \in \mathbb{C}$$

then f is constant.

Solution: They hypothesis implies that for all integers m and n, $f(0) = f(m+n\sqrt{2})$. Since $\sqrt{2}$ is irrational, the numbers $m+n\sqrt{2}$ are distinct as m and n range over the integers. For every choice of n, we can choose m so that $m + n\sqrt{2}$ is in $[0, 1]$. So there are infinitely many of them in $[0, 1]$, and so they have an accumulation point in $[0,1]$. The function $f(z) - f(0)$ has zeros at all these points, so it is the zero function. So $f(z)$ is constant.

3. Let f be analytic on an open set U. Let $z_1, z_2, \cdots z_n$ be the distinct zeroes of f and $m_1, m_2, \cdots, m_n$ their multiplicities. Let γ be a closed contour which is homotopic in U to a point. Compute the following integral in terms of z_j, m_j and the winding numbers $n(\gamma, z_j)$, and justify your computation.

$$\int_\gamma \frac{zf'(z)}{f(z)} dz$$

Solution: Let

$$g(z) = \frac{f(z)}{(z-z_1)^{m_1} \cdots (z-z_n)^{m_n}}$$

Then $g(z)$ is analytic on U except possibly at the z_j. For each j, there is a neighbor of z_j and a non-zero analytic function $h(z)$ on that neighborhood such that $f(z) = (z-z_j)^{m_j} h(z)$ on that neighborhood. So on this neighborhood of z_j,

$$g(z) = \frac{h(z)}{(z-z_1)^{m_1} \cdots (z-z_{j-1})^{m_{j-1}} (z-z_{j+1})^{m_{j+1}} \cdots (z-z_n)^{m_n}}$$

So the singularity at z_j is removable. Thus $g(z)$ is analytic on U. It is also nonzero on U.

Now

$$f(z) = g(z)(z-z_1)^{m_1} \cdots (z-z_n)^{m_n}$$

So

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \sum_{j=1}^{n} \frac{zm_j}{z-z_j}$$

Since γ is homotopic to a point and $zg'(z)/g(z)$ is analytic on U, by Cauchy's theorem its integral along γ is zero. And

$$\int_\gamma \frac{zm_j}{z-z_j} dz = \int_\gamma (m_j + \frac{m_j z_j}{z-z_j}) dz = m_j z_j \int_\gamma \frac{1}{z-z_j} dz = m_j z_j n(\gamma, z_j)$$

Thus the orignal integral given in the problem is

$$2\pi i \sum_{j=1}^{n} m_j z_j n(\gamma, z_j)$$

4. Let f be analytic on a domain U. Define

$$M = \sup_{z \in U} Ref(z)$$

Suppose there is a $z_0 \in U$ with $Ref(z_0) = M$. Prove that f is a constant.

Solution: If f is not constant then by the open mapping theorem f is an open map. Pick $\epsilon > 0$ so that $D_\epsilon(z_0) \subset U$. Then $f(D_\epsilon(z_0))$ is an open set containing $f(z_0)$. So it contains a point with real part greater than M. But this implies there is a $z \in D_\epsilon(z_0)$ with $Ref(z) > M$ which contradicts the definition of M.

5. Let $\log(z)$ be the branch of the log with branch cut along the negative real axis. It has a power series about $z_0 = -1 + i$:

$$log(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

A confused professor claims that since the distance from z_0 to the branch cut is 1, the radius of convergence of the power series is 1. Explain what is wrong with this reasoning and find (with justification) the correct radius of convergence.

Solution: To find the radius of convergence we need to find the largest disc about z_0 on which the function has an analytic continuation. If we take a disc of radius $\sqrt{2}$, then the branch of the log with branch cut along the positive axis provides an analytic continuation. The fact that the branch with the cut along the negative real axis is not analytic on this disc is irrelevant. There is no analytic continuation on a bigger disc, since its derivative would have to be $1/z$ and this is singular at 0.

To check our answer we can compute the power series.

6. Suppose that f is analytic in a disc of radius ρ about 0 and has a zero of order n at 0. Suppose that f has no other zero in this disc. Define a closed contour by $\gamma(t) = f(re^{it})$ for $0 \le t \le 2\pi$, with $r < \rho$. Compute the winding number of γ about 0 in terms of n and r and justify your computation.

Solution: The winding number is

$$\int_\gamma \frac{dz}{z}$$

Since γ is a differentiable curve, this equals

$$\int_0^{2\pi} \frac{\gamma'(t)dt}{\gamma(t)} = \int_0^{2\pi} \frac{f'(re^{it})ire^{it}dt}{f(re^{it})} = \int_{|z|=r} \frac{f'(z)dz}{f(z)}$$

By the argument principle the last integral is $2\pi in$. It does not depend on r.