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1 Riemann surfaces

1.1 Definitions and examples

Let X be a topological space. We want it to look locally like $\mathbb{C}$. So we make the following definition.

Definition 1. *A* **complex chart** *on X is a homeomorphism $\phi : U \to V$ where U is an open set in X and V is an open set in $\mathbb{C}$. We say the chart is centered at $p \in U$ if $\phi(p) = 0$.*

We think of the chart as providing a complex coordinate $z = \phi(p)$ locally on X. Charts are also called local coordinates and sometimes uniformizing variables. When two charts have overlapping domains, these charts need to be related.

Definition 2. *Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two charts. We say they are* **compatible** *if either U_1 and U_2 are disjoint or $\phi_2 \circ \phi_1^{-1}$ is analytic on $\phi_1(U_1 \cap U_2)$.*

Note that if $\phi_2 \circ \phi_1^{-1}$ is analytic on $\phi_1(U_1 \cap U_2)$, then $\phi_1 \circ \phi_2^{-1}$ is analytic on $\phi_2(U_1 \cap U_2)$. So the definition is symmetric in the two charts. We refer to the functions such as $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ as transition *functions*.

Lemma 1. *The derviative of a transition function never vanishes.*

Proof: The derivative of an injective function never vanishes. □

Definition 3. *A* **complex atlas** *$\mathcal{A}$ on X is a collection $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ of compatible charts whose domains cover X, i.e., $X = \cup_\alpha U_\alpha$.*

There will be many atlases on a given Riemann surface.

Definition 4. *Two complex atlases $\mathcal{A}$ and $\mathcal{B}$ are* **equivalent** *if every chart in $\mathcal{A}$ is compatible with every chart in $\mathcal{B}$.*

When two atlases are compatible we can combine them to get another atlas which contains them both. A Zorn's lemma argument shows that every atlas is contained in a unique maximal atlas. Furthermore, two atlases are compatible if and only if they are subcollections of the same maximal atlas.

Definition 5. *A* **complex structure** *on X is a maximal atlas on X. Equivalently it is an equivalence class of complex atlases on X.*

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Finally we come to our main definition.

Definition 6. *A* **Riemann surface** *is a connected second countable Hausdorff topological space with a complex structure.*

Recall that Hausdorff means ... and second countable means that there is a countable basis for the topology. Note that if X is a subset of some $\mathbb{R}^n$ and its topological is the subspace topology, then it is automatically Hausdorff and second countable.

We now consider examples.

1. The **complex plane:** $X = \mathbb{C}$. There is a trivial chart: $U = X$, $V = \mathbb{C}$ with $\phi(z) = z$. There are of course many other charts. (Give some.)

Any connected open subset of $\mathbb{C}$ is a Riemann surface. Recall that all simply connected domains that are not all of $\mathbb{C}$ are conformally equivalent. We will see that they are also "equivalent" (yet to be defined) Riemann surfaces. We focus on one of them.

2. The **upper half plane** $X = \mathbb{H}$. Again, there is a single trivial chart that covers all of X.

3. **The Riemann sphere:** $\hat{\mathbb{C}}$. This is the first example that cannot be covered by a single chart. Let

$$S^2 = \{(x, y, w) \in \mathbb{R}^3 : x^2 + y^2 + w^2 = 1\} \tag{1}$$

be the unit sphere in $\mathbb{R}^3$. We give it the topology it it inherits as subspace of $\mathbb{R}^3$. Let $\phi_1 : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$ be defined by

$$\phi_1(x, y, w) = \frac{x}{1-w} + i\frac{y}{1-w} \tag{2}$$

and $\phi_2 : S^2 \setminus \{(0, 0, -1)\} \to \mathbb{C}$ be defined by

$$\phi_2(x, y, w) = \frac{x}{1+w} - i\frac{y}{1+w} \tag{3}$$

These maps are just stereographic projections with respect to the north and south poles. You should check this. Note that I am using a slightly different definition of stereographic projection from last semester. One of the homework problems will be to compute $\phi_2 \circ \phi_1^{-1}$ and check that it is holomorphic.

4. **Tori:** Fix two complex numbers ω_1, ω_2 which are linearly independent over $\mathbb{R}$. Define the lattice

$$L = \{m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z}\} \tag{4}$$

Then define $X = \mathbb{C}/L$ and let $\pi : \mathbb{C} \to X$ be the projection map. We give X the quotient space topology, i.e., $U \subset X$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{C}$. The makes X a connected, compact topological space. Next we define some charts. Pick $\epsilon > 0$ small enough that $|\omega| > 2\epsilon$ for all nonzero $\omega \in L$. Now let $z \in \mathbb{C}$ and consider the disc $B_\epsilon(z)$. The choice of ϵ insures that no two points in $B_\epsilon(z)$ differ by a nonzero lattice point. So π restricted to the disc is 1-1 and so defines a homeomorphism between $B_\epsilon(z)$ and $\pi(B_\epsilon(z))$. Letting z range over $\mathbb{C}$ gives our family of charts. Clearly the $\pi(B_\epsilon(z))$ cover X.

We need to show these charts are compatible. Let $z_1, z_2 \in \mathbb{C}$ and let ϕ_1, ϕ_2 be the charts that map $\pi(B_\epsilon(z_i))$ onto $B_\epsilon(z_i)$. Suppose that $\pi(B_\epsilon(z_1))$ and $\pi(B_\epsilon(z_2))$ overlap. Let $T(z) = \phi_2(\phi_1^{-1}(z))$. We have $\pi(T(z)) = \pi(z)$ on the overlap. So $T(z)$ and z must differ by a lattice element: $T(z) = z + \omega$. Since $T(z) - z$ is continuous and the lattice is discrete and the overlap is connected, ω must be same for all z in the overlap. So $T(z)$ is just $z + c$ for a constant c and so is analytic.

5. The **complex projective line** $\mathbb{CP}^1$. For a nonzero zero vector $(z, w) \in \mathbb{C}^2$ we let $[z : w]$ denote the span of (z, w), i.e., the set of $(\lambda z, \lambda w)$ where λ ranges over nonzero complex numbers. $\mathbb{CP}^1$ is the set of $[z : w]$. Let $U_0 = \{[z : w] : z \neq 0\}$, and on this set define $\phi_0([z : w]) = w/z$. Similary $U_1 = \{[z : w] : w \neq 0\}$, and $\phi_1([z : w]) = z/w$. We leave it to the homework to check that these two charts are comaptible.

We now look at maps between Riemann surfaces.

Definition 7. *Let M and N be Riemann surfaces and $f : M \to N$ a continuous map between them. We say f is holomorphic or analytic if for every chart $\{U, \phi\}$ on M and every chart $\{V, \psi\}$ on N with $U \cap f^{-1}(V) \neq \emptyset$, the function $\psi \circ f \circ \phi^{-1}$ is holomorphic on the open subset of $\mathbb{C}$ where it is defined. (Note that this is map from an open subset of $\mathbb{C}$ into $\mathbb{C}$, so it makes sense to say it is holomorphic.) The map f is said to be conformal if it is holomorphic and a bijection. In this case the inverse is also holomorphic. When two Riemann surfaces have a conformal map between them, we say they are isomorphic (as Riemann surfaces).*

If N is just $\mathbb{C}$, then we call f a holomorphic function. If N is $\hat{\mathbb{C}}$ and f is not identically ∞, then we call f a meromorphic function. We denote the $\mathbb{C}$-algebra of holomophic functions on M by $\mathcal{H}(M)$, and the $\mathbb{C}$-algebra of meromorphic functions on M by $\mathcal{K}(M)$.

Proposition 1. *A non-constant holomorphic map is an open map.*

Proof. Homework. ☐

Proposition 2. *Let f be an analytic map between Riemann surfaces, $f : M \to N$. Let $q \in N$ and suppose there exist a sequence $p_n \in M$ which converges to some $p \in M$ and for which $f(p_n) = q$. Then f is constant.*

Proof. Homework. ☐

Theorem 1. *Let M be a compact Riemann surface and N a Riemann surface. Let $f : M \to N$ be a holomorphic mapping. Then f is either constant or surjective. In particular the only holomorphic functions on a compact Riemann surfaces are the constants.*

Proof. Suppose f is not constant. Than it is an open map. So $f(M)$ is open being the image of the open set M. But $f(M)$ is also the image of a compact set under a continuous function, so $f(M)$ is compact. In a Hausdorff space compact implies closed. Thus $f(M)$ is both open and closed. Since N is connected, $f(M)$ must be all of N. ☐

Let $f : M \to N$ be a holomorphic map between two Riemann surfaces. Fix a point $p_o \in M$. Let ϕ be a chart for M centered at p_0 and ψ a chart for N centered by $f(p_0)$. Then $\psi \circ f \circ \phi^{-1}(z)$ is analytic in a neighbor of $z = 0$ and vanishes at $z = 0$. Let n be the order of the zero. Then we can write this function as $z^n h(z)$ where h does not vanish in a neighborhood of 0. Furthermore, we can find an analytic function $g(z)$ such that $h(z) = g(z)^n$ in a neighborhood of zero. So $\psi \circ f \circ \phi^{-1}(z) = [zg(z)]^n$.

We will now show that we can change charts so that this function just becomes z^n. Let $\tilde{\phi}(p) = \phi(p)g(\phi(p))$. This is a chart in some neighborhood of p. We have $\tilde{\phi} \circ \phi^{-1}(z) = zg(z)$, so $\psi \circ f \circ \phi^{-1}(z) = [\tilde{\phi} \circ \phi(z)]^n$. which leads to $\psi \circ f \circ \tilde{\phi}^{-1}(z) = z^n$.

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Definition 8. *We say that f has multiplicity n at p_0 and that n is the* **ramification number** *of f at p_0. The* **branch number** *of f at p_0 is $b_f(p_0) = n - 1$.*

An analytic function on a domain in $\mathbb{C}$ has a zero of order greater than one if and only its derivative vanishes at the zero. Since zeros of analytic functions are isolated, it follows that the the points with ramification number greater than 1 are isolated. If M is compact this implies there are only finitely many of them. The following proposition says that the number of times a point in N is hit is the same for all points in N.

Proposition 3. *Let $f : M \to N$ be a non constant holomophic map between two compact Riemann surfaces. Then there is an integer m such that every $q \in N$ is assumed exactly m times (counted according to multiplicty) by f, i.e.,*

$$\sum_{p \in f^{-1}(q)} (b_f(p) + 1) = m, \quad \forall q \in N \tag{5}$$

Definition 9. *We call m the degree of f and write it as $m = deg f$. We also say f is an m-sheeted covering of N by M.*

Proof. Define

$$N_k = \{q \in N : \sum_{p \in f^{-1}(q)} (b_f(p) + 1) \geq k\} \tag{6}$$

Obviously, $N_{k+1} \subset N_k$. We will show that each N_k is either empty or all of N. Then m is the largest k for which N_k is all of N. Since N is connected, to show N_k is empty or N it suffices to show it is both open and closed.

N_k is open: Let $q_0 \in N_k$. Consider a $p_0 \in f^{-1}(q_0)$. We can find charts at p_0 and q_0 so that f is just z^n where n is the ramification number at p_0. So for q near q_0 there are n points near p_0 that get mapped to q. Summing over $p_0 \in f^{-1}(q_0)$, we see

$$\sum_{p \in f^{-1}(q)} (b_f(p) + 1) \geq \sum_{p_0 \in f^{-1}(q_0)} (b_f(p_0) + 1) \geq k \tag{7}$$

for q near q_0, i.e., there is a neighborhood of q_0 contained in N_k.

N_k is closed: Let $q_n \in N_k$ with $q_n \to q$. We can assume the q_n are distinct. There are only a finite number of points with ramification number greater than 1, so we can assume that all the q_n have ramification number 1. So $q_n \in N_k$ implies there are k distinct points that get mapped to q_n. Label them $p_n^1, \cdots, p_n^k$. Since M is compact, a diagonalization argument shows there is a subsequence n_j of n such that for each l, $p_{n_j}^l$ converges. Let p^l be its limit. We have $f(p^l) = q$. If the p^l are all distinct, this shows $q \in N_k$. If a group of m of the p^l are the same, then by working in coordinates we can show the ramification number of that point is m. See proposition below. ☐

Proposition 4. *Let f be analytic in a neighborhood of 0. Let $w_n \in \mathbb{C}$ with $w_n \to 0$. For each n, let $z_n^1, \cdots, z_n^m$ be distinct complex numbers in the domain of f such that $f(z_n^j) = w_n$ for $j = 1, \cdots, m$. Suppose also that for $j = 1, \cdots, m$, $\lim_{n\to\infty} z_n^j = 0$. Then z is a zero of f with multiplicity at least m.*

Corollary 1. *A meromorphic function on compact Riemann surface has the same number of zeroes and poles.*

1.2 Topology of Riemann surfaces

One nice consequence of the existence of a meromorphic function is an easy proof of the existence of a triangulation for compact surfaces.

Definition 10. *Let S be a compact surface. A triangulation is a finite number of closed sets $T_1, T_2, \cdots, T_n$ in S and homeomorphisms ϕ_i of T_i to $\mathbb{R}^2$ such that $\phi_i(T_i)$ is a closed triangle in $\mathbb{R}^2$. For $i \neq j$, the triangles $\phi_i(T_i)$ and $\phi_j(T_j)$ must either be disjoint, have single vertex in common or share one edge. (The edge of one triangle cannot be a proper subset of the edge of another triangle.)*

A compact surface has a triangulation, but the proof is not trivial. For a Riemann surface M we can prove it as follows. Let f be a meromorphic function on M, i.e., a holomorphic function onto $\hat{\mathbb{C}}$. Construct a triangulation $T_1, \cdots, T_n$ of $\hat{\mathbb{C}}$ such that the images of the ramified points in M under f are at vertices in triangulation and such that on each component of $f^{-1}(int\, T_i)$is injective.

Riemann surfaces are always orientable, so in the following review we only consider orientable, triangulable compact surfaces M.

1.2.1 Homotopy

Let $P, Q \in M$. Let $c_i : [0,1] \to M$ be curves from P to Q for $i = 1, 2$. There are homotopic if there is continuous $h : [0,1] \times [0,1] \to M$ such that $h(\cdot, 0)$ is c_1, $h(\cdot, 1)$ is c_2, and for each s, $h(s, \cdot)$ is a curve from P to Q. Now fix a $P \in M$. We consider curves that start and end at P. We define two such curves to be equivalent if they are homotopic. It is not hard to check it an equivalent relation. The equivalence classes are the elements of the first homotopy group $\pi_1(M)$. The product of two curves c_1 and c_2 is the curve we get by first traversing c_1 and then c_2. The inverse of a curve is the same curve traversed in the opposite direction. The identity is the curve that is just the single point P. Of course, one has to check that these definitions are well defined for the equivalence classes. Since M is connected (and so path-wise connected), it is not hard to show that the homotopy group based at P is isomophic to that based at another point in M. This common group is $\pi_1(M)$.

Examples

1. $\pi_1(S^1) = \mathbb{Z}$
2. $\pi_1(S^2) = \pi_1(\hat{\mathbb{C}}) = \{0\}$
3. $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$

1.3 Differential forms

We assume that the reader has seen the theory of integration on differentiable manifolds. A Riemann surface is a two dimensional real manifold. So all that theory applies here. Throughout this section M is a Riemann surface. We will use z to denote a complex chart. This is a complex valued function on an open set in M, so we can write it as $z = x + iy$ where x and y are real values functions on the open set in M.

Definition 11. *A* 0**-form** *is a continuous function on M.*

Definition 12. *A* 1**-form** *ω is an ordered assignment of continuous complex valued functions f and g to each complex chart on M (f and g are defined on the domain of the chart). We write $\omega = f dx + g dy$. The assignment must "transform correctly under coordinate changes." This means that if $\tilde{z} = \tilde{x} + i\tilde{y}$ is another complex chart that overlaps z and $\omega = \tilde{f} d\tilde{x} + \tilde{g} d\tilde{y}$ then*

$$\begin{pmatrix} \tilde{f}(\tilde{z}) \\ \tilde{g}(\tilde{z}) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \tilde{x}} & \frac{\partial y}{\partial \tilde{x}} \\ \frac{\partial x}{\partial \tilde{y}} & \frac{\partial y}{\partial \tilde{y}} \end{pmatrix} \begin{pmatrix} f(z(\tilde{z})) \\ g(z(\tilde{z})) \end{pmatrix} \tag{8}$$

Definition 13. *A* 2-**form** ω *is an assignment of a continuous complex valued functions* f *to each complex chart on* M *We write* $\omega = f dx \wedge dy$*. The assignment must "transform correctly under coordinate changes." This means that if* $\tilde{z} = \tilde{x} + i\tilde{y}$ *is another complex chart that overlaps* z *and* $\omega = \tilde{f} d\tilde{x} \wedge d\tilde{y}$*, then*

$$\tilde{f}(\tilde{z}) = \frac{\partial(x,y)}{\partial(\tilde{x},\tilde{y})} f(z(\tilde{z}))$$

where $\frac{\partial(x,y)}{\partial(\tilde{x},\tilde{y})}$ *is the determinant of the two by two Jacobian that appeared in the previous def.*

Recall how exterior multiplication (wedge product) of forms works: $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$. In general you can take the wedge product to a k-form and an l-form to get a $k + l$ form. But since we are on a two dimensional manifold, we will only see wedge products of two 1-forms.

Next we recall integration. A 1-form can be integrated along a curve in the manifold. If the curve lies in a single chart, we let $\gamma : [a, b] \to M$ be the curve. Let z be the chart, so $z(\gamma(t))$ are the coordinates of the curve. Then the integral is given by

$$\int_\gamma \omega = \int_a^b \left[f(z(\gamma(t))) \frac{dx}{dt} + g(z(\gamma(t))) \frac{dy}{dt} \right] dt \tag{9}$$

The transition formula for change of coords for 1-forms implies this is independent of the choice of chart. If the curve lies in more than one chart, we break it up into pieces each of which lies in a single chart.

We can integrate a two form Ω over subsets of the Riemann surface. Let D be a subset and suppose it is contained in a single chart $\{z, U\}$. Then for the two form $\Omega = f(x, y) dx \wedge dy$,

$$\int\int_D \Omega = \int\int_{z(U)} f(x,y) dxdy \tag{10}$$

For a subset that is not contained in a single chart, write it as a union of subsets that are. (partitions of unity).

Curves are 1-chains and surfaces are 2-chains. 0-chains are sets of points. The integral of a 0-form (a function f) over a 0-chain is just the sum of the values of the function at the points.

The d operator: For a differentiable function f we define a 1-form by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{11}$$

where $z = x + iy$ is a complex chart. For a 1-form $\omega = fdx + gdy$ we define

$$d\omega = df \wedge dx + dg \wedge dy = \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dx \wedge dy \tag{12}$$

Since we are on a two dimensional manifold, for a 2-form Ω, $d\Omega = 0$.

In general, Stoke's theorem says that for a k form ω and a $k+1$ chain c,

$$\int_c d\omega = \int_{\partial c} \omega \tag{13}$$

In our setting, k can only be 0 or 1. The case of $k = 0$ is just the fundamental theorem of calc. In the case $k = 1$, the left side is a surface integral and the right side is an integral along the curve that bounds the surface.

Let f be differentiable on the Riemann surface. Given coordinates $z = x + iy$, we can think of f as a complex valued function of (x, y). We let f_x and f_y denote the partial derivatives with respect to x and y.

Definition 14. *For a C^1 function f on the Riemann surface and coordinates $z = x + iy$,*

$$\begin{aligned}
f_z &= \frac{1}{2}(f_x - if_y) \\
f_{\overline{z}} &= \frac{1}{2}(f_x + if_y) \\
dz &= dx + idy, \quad d\overline{z} = dx - idy \\
\partial f &= f_z dz, \quad \overline{\partial} f = f_{\overline{z}} d\overline{z},
\end{aligned}$$

For an analytic function, $f' = f_x = -if_y$, so $f_z = f'$ and the CR equations are equivalent to $f_{\overline{z}} = 0$.

Lemma 2. *∂f and $\overline{\partial} f$ are 1-forms.*

$$\begin{aligned}
d &= \partial + \overline{\partial} \\
\partial^2 &= \overline{\partial}^2 = \partial\overline{\partial} = \overline{\partial}\partial = 0 \\
dz \wedge d\tilde{z} &= -2idx \wedge dy
\end{aligned}$$

We can write a 1-form as $u(z)dz+v(z)d\tilde{z}$ where for every complex chart, u and v are complex valued functions. And we can write a 2-form as $g(z)dz\wedge d\tilde{z}$ for a complex valued funciton $g(z)$. So far we have only used the complex notation to rewrite some things. We have not really used the complex structure. Now we will.

Definition 15. *Let ω be a 1-form. Given a chart $z = x + iy$ and $\omega = fdx + gdy$, we define a new 1-form, the conjugate of ω, by*

$$*\omega = -gdx + fdy$$

Note that $**\omega = -\omega$.

Proposition 5. *The above definition is well defined and $*\omega$ is a 1-form, i.e., it transforms correctly under coordinate changes. If $\omega = udz + vd\overline{z}$, then $*\omega = -iudz + ivd\overline{z}$*

Proof. The proof of the first sentence is a homework problem. The second sentence is a trivial calculation. ☐

Definition 16. *Let ω be a 1-form on M. It is* **exact** *if there is a C^1 function f on M with $\omega = df$. It is* **closed** *if it is C^1 and $d\omega = 0$. ω is* **co-exact** *if $*\omega$ is exact and* **co-closed** *if $*\omega$ is closed.*

Every exact form is closed and every co-exact form is co-exact. The converses are only true locally unless M is simply connected in which case they are true globally.

Definition 17. *Let f be a C^2 function on the Riemann surface M. The Laplacian of f is a two form defined by*

$$\Delta f = (f_{xx} + f_{yy})dx \wedge dy$$

We say f is **harmonic** *if $\Delta f = 0$. This is a local property. A 1-form is* **harmonic** *if it is locally given by df where f is a harmonic function (locally).*

Of course we should check that this is independent of the choice of coordinates. This is part of a homework problem.

Lemma 3. *Let f be C^2 on M. Then*

$$\Delta f = d * df = -2i\partial\overline{\partial} f$$

Proposition 6. *A 1-form is harmonic if and only if it is closed and co-closed*

Proof. Let ω be a harmonic 1-form. Locally it is exact and so is closed. It is co-closed by the above lemma.

Now let ω be closed and co-closed. Closed implies locally it is exact, $\omega = df$ for some f. Since it is co-closed, $\Delta f = 0$ by the above lemma. □

Definition 18. *A 1-form is* **holomorphic** *if locally it can be written as $\omega = df$ where f is holomorphic.*

Proposition 7. *(i) ω is holomorphic if and only if for all coordinates z, when we write $\omega = udz + vd\overline{z}$, then $v = 0$ and u is holomorphic in z.*

(ii) If u is a harmonic function, then ∂u is a holomorphic 1-form.

(iii) A holomorphic 1-form is harmonic.

(iv) A 1-form is holomorphic if and only if there is a harmonic 1-form α such that $\omega = \alpha + i * \alpha$.

(v) A 1-form ω is holomorphic if and only if it is closed and $*\omega = -i\omega$

Proof. (i) Suppose ω is holomorphic. Then locally, $\omega = df$ where f is holomorphic. So $\omega = (\partial + \overline{\partial})f = f_z dz + f_{\overline{z}} d\overline{z}$. Since f is holomorphic, $f_{\overline{z}} = 0$ and f_z is a holomorphic function.

Now suppose for all coordinates z, $\omega = udz$ with u holomorphic. Then locally u has a primitive, i.e., there is a holomorphic function g such that $u = g_z$ locally. Then $dg = g_z dz + g_{\overline{z}} d\overline{z} = udz = \omega$.

(ii) Let u be a harmonic function. So $\overline{\partial}\partial u = 0$. Since $\partial u = u_z dz$, $\overline{\partial} u_z = 0$, i.e., $u_{z,\overline{z}} = 0$. So u_z is holomorphic. By (i), ∂u is a holomorphic 1-form.

(iii) Let ω be holomorphic. Then $\omega = udz$ with $u_{\overline{z}} = 0$. By previous proposition to show ω is harmonic it suffice to show it is closed and co-closed which is immediate.

(iv) Suppose α is a harmonic 1-form. Then it is closed and co-closed. So

$$\begin{aligned} 0 &= d\omega = (u_{\overline{z}} - v_z) d\overline{z} \wedge dz \\ 0 &= d * \omega = -i(u_{\overline{z}} + v_z) d\overline{z} \wedge dz \end{aligned}$$

Hence $u_{\overline{z}} = 0 = v_z$. So u and $\overline{v}$ are holomorphic functions. So udz and $\overline{v}dz$ are holomorphic 1-forms. So $\alpha = \omega_1 + \overline{\omega_2}$ where ω_1, ω_2 are holomorphic 1-forms. Since $\alpha + i * \alpha = 2\omega_1$, this shows $\alpha + i * \alpha$ is holomorphic.

Now let ω be holomorphic. Define $\alpha = (\omega - \overline{\omega})/2$. Since ω and $\overline{\omega}$ are harmonic, α is too. Calculation shows $\alpha + i * \alpha = \omega$.

(v) Let ω be holomorphic. Then it is harmonic and so is closed.

Now suppose ω is closed and $*\omega = -i\omega$. If we write $\omega = udz + vd\overline{z}$, then $*\omega = -i\omega$ implies $\omega = udz$. The fact that ω is closed implies $u_{\overline{z}} = 0$, i.e., u is holomorphic. □

1.4 More examples

6. **graphs of analytic functions:** This is a trivial example, but serves to motivate the next example. Let $V \subset \mathbb{C}$ be a domain. Let $g(z)$ be holomorphic on V. Let X be the graph of g. This is the subset of $\mathbb{C}^2$ given by

$$X = \{(z, g(z)) : z \in V\}$$

We give this the subspace topology from $\mathbb{C}^2$. We can cover it with a single chart. Define π on X by $\pi((z, g(z)) = z$. Clearly this Riemann surface is isomorphic to V.

7. **Smooth affine plane curves**

Instead of looking at a graph $w = g(z)$, we look at a curve that is defined implicitly $g(z, w) = 0$. We need a "calculus" result :

Theorem 2. *(Implicit function theorem) Let $f(z, w)$ be a polynomial in z and w. Let*

$$X = \{(z, w) : f(z, w) = 0\}$$

Let $p_0 = (z_0, w_0) \in X$. Suppose $\frac{\partial f}{\partial w}(p_0) \neq 0$ Then there is a holomorphic function $g(z)$ defined on a neighborhood of z_0 such that there is a neighborhood U of p_0 for which $U \cap X$ is the graph of g. Moreover, near z_0, $g' = -\frac{\partial f}{\partial z}/\frac{\partial f}{\partial w}$.

We use the theorem to make X into a Riemann surface. The polynomial f is non-singular if for every root (z, w) at least one of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial w}$ is non-zero. The graph of a non-singular polynomial is a smooth affine curve.

If $p_0 = (z_0, w_0) \in X$ with $\frac{\partial f}{\partial w}(p_0) \neq 0$, then we apply the implicit function theorem to get a holomorphic $g(z)$ on a neighborhood U of z_0 such that X near p_0 is the graph of g. The projection $\pi_z : (z, g(z)) = z$ then defines a homomorphism of a neighborhood of p_0 to U. The assumption of nonsingular insures that these charts cover X. We need to check they are compatible. If two overlapping charts both use projection with respect to the z variable, then the transition function if just the identity. Likewise if they both use projection with respect to the z variable. The nontrivial case is when one chart uses z and the other w. In this case the transition function is just g, and so is analytic. (sketchy).

Recall that part of our definition of a Riemann surface is that it be connected. For a general polynomial the graph need not be connected. We can consider its connected components invidually and they are Riemann surfaces. A non-trivial theorem from algebraic geometry says that if $f(z, w)$ is irreducible, then then X is connected.

Note that these Riemann surfaces will never be compact. For each w, $f(z, w)$ is a polynomial in z and so has at least one zero.

8. **Projective curves**

We start by defining the projective plane $\mathbb{P}^2$. The construction is analogous to that for the projective line. $\mathbb{P}^2$ is the set of one dimensional subspaces (over $\mathbb{C}$) of $\mathbb{C}^3$. We let $[x : y : z]$ denote the 1-d subspace spanned by a nonzero vector (x, y, z) in $\mathbb{C}^3$. Of course, the elements of this subspace are of the form $(\lambda x, \lambda y, \lambda z)$ where $\lambda \in \mathbb{C}$. We have the set $\mathbb{P}^2$ into a two dimensional complex manifold as follows. Let

$$U_1 = \{[x : y : z] : x \neq 0\}, U_2 = \{[x : y : z] : y \neq 0\}, U_3 = \{[x : y : z] : z \neq 0\}$$

On U_1 we define $\phi_1([x, y, z]) = (y/x, z/x)$. This is a homeomorphism of U_1 onto $\mathbb{C}^2$. The definitions of ϕ_2 and ϕ_3 on U_2 and U_3 are analogous. Of course, $\mathbb{P}^2$ is not a Riemann surface. It is compact. (Prove this!). So we can get compact Riemann surfaces by looking at zero sets in this space.

Let $f(x, y, z)$ be a polynomial in three complex variables. We say that it is homogeneous of degree d if

$$f(\lambda x, \lambda y, \lambda z) = \lambda^d f(x, y, z)$$

Note that for such a polynomial, the statement $f([x : y : z]) = 0$ is well defined. This defines a subset X of $\mathbb{P}^2$. Let $X_i = X \cap U_i = \{[x : y : z] : x \neq 0, F(x, y, z) = 0\}$. This is homeomorphic to $\{(y, z) \in \mathbb{C}^2 : F(1, y, z) = 0\}$. This last set is a affine plane curve. Is it smooth? Of course we need a condition on F.

Definition 19. *A homogeneous polynomial F is nonsingular if there is no nonzero solution to the system of equations*

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

Lemma 4. *Let $F(z, y, z)$ be a homogeneous polynomial. Then F is nonsingular if and only if each X_i is a smooth affine plane curve.*

Proof. We only prove one direction: F nonsingular implies that each X_i is smooth. Suppose X_1 is not smooth. Then $f(y, z) = F(1, y, z)$ is singular, i.e., there is a (y_0, z_0) at which both $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ vanish and $f(y_0, z_0) = 0$. Thus $F(1, y_0, z_0) = 0$. Now

$$\begin{aligned} \frac{\partial F}{\partial x}(1, y_0, z_0) &= \frac{\partial f}{\partial x}(y_0, z_0) = 0 \\ \frac{\partial F}{\partial y}(1, y_0, z_0) &= \frac{\partial f}{\partial y}(y_0, z_0) = 0 \end{aligned}$$

If d is the degree of F, then

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = dF$$

which shows that

$$\frac{\partial F}{\partial z}(1, y_0, z_0) = 0$$

□

So if F is a nonsingular homogeneous polynomial, then each X_i is a Riemann surface. The charts for the X_i provide charts that cover X. To complete the proof that X is a Riemann suface we have to show that charts coming from different X_i are compatible. We leave this to the reader.