February 12, 2010

2 Existence theorems

2.1 Hilbert space review

I include here some notes on Hilberts spaces including a proof of the projection theorem. They are notes from 523a and the notation is a bit different.

Normed linear spaces are vector spaces with a norm. When they are complete, i.e., every Cauchy sequence converges, we call them Banach spaces. A Hilbert space is a vector space with an inner product which is complete in the norm that comes from the inner product. I will only consider Hilbert spaces which are vector spaces over the real or complex numbers. The properties of the inner product are

Definition: Let H be a vector space over $\mathbb{R}$ or C. An inner product on H is a function

$$(\quad , \quad) : H \times H \to F$$

where F is either R or C such that
(a) $(x, y) = \overline{(y, x)}$
(b) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$
(c) $(x, x) \geq 0$
(d) $(x, x) = 0$ if and only if $x = 0$.
In the above, $x, y, z \in H$ and α, β are real or complex numbers.

In the real case property (a) just says that $(x, y) = (y, x)$. Properties (a) and (b) imply that in the complex case

$$(z, \alpha x + \beta y) = \overline{\alpha}(z, x) + \overline{\beta}(z, y)$$

We say that the inner product is linear in the first argument and anti-linear in the second argument. It is just a convention that we do things this way. We could just as well make the convention that the inner product is linear in the second argument and anti-linear in the first argument. Unfortunately, both conventions are used. I am using the convention that mathematicians use, but physicists use the other convention.

Given an inner product we can define a norm by

$$||x|| = \sqrt{(x, x)}$$

By (c) the quantity inside the square root is negative.

Proposition H.1.1: The norm defined above is indeed a norm. In particular we have the triangle inequality

$$||x + y|| \leq ||x|| + ||y||$$

We also have the the Cauchy-Schwarz inequality

$$|(x, y)| \leq ||x||||y||$$

Furthermore, we have the parallelogram identity

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

Remark: Draw a parallelogram with vertices at $0, x, y$, and $x+y$. Then the parallelogram identity says the sum of the squares of the lengths of the four sides is related to the sum of the squares of the lengths of the diagonals. This is not true in arbitrary normed linear spaces. In fact, if you have a norm for which the parallelogram identity holds, then you can define an inner product so that $||x|| = \sqrt{(x, x)}$.

Examples 1. R^n and C^n with the usual inner product or "dot product"

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}$$

are Hilbert spaces.
2. Let l^2 denote the set of sequences $(x_n)_{n=1}^{\infty}$ which are square summable, i.e., $\sum_n |x_n|^2 < \infty$. If we take real sequences we will have a Hilbert space over R, and if we take complex sequences we will have one over C. If $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ are two such sequences then their inner product is

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

This is a Hilbert space. We proved this space was complete in class.

3. Consider the set of continuous functions on $[0, 1]$, $C([0, 1])$. We can define an inner product on it by

$$(f, g) = \int_0^1 f(x)\overline{g(x)}dx$$

This is a valid inner product, but the set of continous functions is not complete in this inner product, i.e., there are sequences which are Cauchy in the norm that comes from this inner product, but do not converge in norm to a continuous function. By abstract nonsense every metric space has a completion. We can define $L^2([0, 1])$ as this completion. If you look at how you actually construct this completion, then L^2 is a set of equivalence classes of Cauchy sequences. Yuk. Luckily there is a much more concrete realization of $L^2([0, 1])$ - the set of functions on $[0, 1]$ whose square is Lebesgue integrable. Thus one motivation for studying the Lebesgue integral is that it gives a concrete representation of the completion of the space of continuous functions in this norm.

Hilbert spaces are vector spaces, but they are much more. The inner product lets us talk about *angles*. In particular it makes sense to say that two elements of the Hilbert space are at right angles or orthogonal.

Definition: We say that x and y are orthogonal if $(x, y) = 0$. We denote this property by $x \perp y$.

More generally we can define the angle θ between two elements of the Hilbert space x and y by the equation

$$\cos(\theta) = \frac{(x, y)}{||x||||y||}$$

The following definition will play an important role in the theory.

Definition: Let M be a subspace of the Hilbert space. The orthogonal complement of M, denoted $M^\perp$, is

$$M^\perp = \{x : (x, y) = 0 \quad for \quad all \quad y \in M\}$$

In this definition M can be any subset of the space. In particular it need not be a subspace. For any subset M, $M^\perp$ will be a subspace. In fact we have

Proposition H.1.2: Let M be a subset of the Hilbert space. Then $M^\perp$ is a closed subspace. If M itself is a closed subspace then $(M^\perp)^\perp = M$.

Proof: This will be a homework problem.

Projection theorem: Let M be a closed subspace. Then for every x in the Hilbert space there exist unique vectors $y \in M$ and $z \in M^\perp$ such that $x = y + z$. Furthermore,

$$||x||^2 = ||y||^2 + ||z||^2 \tag{1.1}$$

and $||x-y||$ is equal to the distance from x to M. (Recall that this is defined to be the inf of the distances $||x-w||$ where w ranges over M.)

Terminology: y is called the projection or orthogonal projection of x onto M. Let P be the map which sends x to y. P is called the orthogonal projection onto the subspace M. It follows from the theorem that it is a linear operator.

Application: Suppose we want to approximate functions in $L^2([0,1])$ by tenth degree polynomials. First, we should understand that this is not a precise question yet. We have to decide how we will measure the "goodness" of an approximation. Suppose we decide that we want the tenth degree polynomial $p(x)$ that best approximates $f(x)$ in the sense that the L^2 norm of $f(x) - p(x)$ is minimized. The set of polynomials with degree less than or equal to ten is a subspace of the Hilbert space. It is a closed subspace, but this is not trivial and we will not show it here. Let P be the orthogonal projection onto this subspace. We claim that the best approximation to $f(x)$ is in fact Pf. To see this, note that the projection theorem says that $||f - Pf||_2$ is equal to the distance from f to M. The definition of the distance from f to M is

$$dist(f, M) = \inf_{p \in M} ||f - p||_2$$

So $||f - Pf||_2 \leq ||f-p||_2$ for all $p \in M$. The practical person will want to know how we compute Pf. This will have to wait until we have learned about bases.

We now turn to the proof of of the projection theorem. We will use the following proposition, but first we need a definition.

Definition: A subset E is convex if $x, y \in E$ and $0 \leq t \leq 1$ implies $tx + (1-t)y \in E$.

Proposition H.1.2: Let E be a nonempty closed convex subset. Then there is a unique x in E which is closest to the origin, i.e., $||x|| \leq ||y||$ for all $y \in E$.

Proof of the projection theorem: Fix an x. Let

$$x + M = \{x + w : w \in M\}$$

So $x + M$ is just M shifted by x. Think of M as a plane through the origin. Then $x + M$ is another plane which will not go through the origin in general. (It will go through the origin only if $-x \in M$, or equivalently, $x \in M$.) It is easy to see that $x + M$ is closed (since M is) and convex (since M is a subspace). By the previous proposition $x + M$ contains a unique element of minimal norm. Define z to be this element. Then we define y to be $x - z$. Of course we automatically have $x = y + z$. Since $z \in x + M$, $z = x + w$ for some $w \in M$. Thus $x = y + x + w$. So $y = -w$ and this shows that $y \in M$. We need to show that $z \in M^{\perp}$.

It is enough to show that $(z, w) = 0$ for every $w \in M$ with $||w|| = 1$. Recall that z is the element of $x + M$ with the smallest norm. For any scalar c, $z - cw$ is another element of $x + M$. So $||z|| \leq ||z - cw||$, for all scalars c. This implies

$$(z, z) \leq (z, z) + |c|^2 - \overline{c}(z, w) - c(w, z)$$

and so

$$0 \leq |c|^2 - \overline{c}(z, w) - c(w, z)$$

Now take $c = (z, w)$ and we get that $0 \leq -|c|^2$, and so $c = 0$. Thus $z \in M^{\perp}$.

Property (1.1) follows from $x = y + z$ and the fact that y and z are orthogonal. By the definition of z, $||x - y|| = ||z||$ is the distance from the origin to $x + M$, i.e., the inf of $||x + w||$ as w ranges over M. This is the same as the inf of $||x - w||$ as w ranges over M since M is a subspace. But this last inf is the definition of the distance from x to M.

The uniqueness part of the theorem is left to the reader. □

Proof of proposition H.1.2: Define

$$d = \inf_{y \in E} ||y||$$

We have to show that there is an $x \in E$ with $||x|| = d$ and that it is unique. Let $y, z \in E$. By the parallelogram identity

$$||y-z||^2 + ||y+z||^2 = 2||y||^2 + 2||z||^2$$

Since E is convex, $(y+z)/2 \in E$. Hence

$$||(y+z)/2|| \geq d$$

Thus we have

$$||y-z||^2 \leq 2||x||^2 + 2||y||^2 - 4d^2$$

If y and z both satisfy $||y|| = d$ and $||z|| = d$, then the above implies $||y-z|| = 0$. This proves uniquesness.

Now let x_n be a sequence in E with $||x_n|| \to d$. By the above

$$||x_n - x_k||^2 \leq 2||x_n||^2 + 2||x_k||^2 - 4d^2$$

As n and k go to ∞ the right side converges to 0. Thus x_n is a Cauchy sequence. Since Hilbert spaces are complete we can let x be the limit of this Cauchy sequence. If $||x_n - x|| \to 0$, then by the triangle inequality it follows that $||x_n|| \to ||x||$. But we chose the x_n so that $||x_n|| \to d$, so $||x|| = d$. $\square$

2.2 Weyl's Lemma

Theorem 1. (Weyl's Lemma) *Let ϕ be a measurable square integrable function on the unit disc $\mathbb{D}$. The function ϕ is C^2 and harmonic if and only if*

$$\int\int_{\mathbb{D}} \phi \Delta \eta = 0 \tag{1}$$

for all C^∞ functions η on $\mathbb{D}$ with compact support.

Proof: The implication that ϕ harmonic implies (1) is fairly short and was proved by Prof. Wehr.

If ϕ is C^2, then showing that (1) implies that ϕ is harmonic is also fairly short and was proved by Prof. Wehr.

The real heart of the theorem is showing that if ϕ satisfies (1), then ϕ is C^2. The general strategy is to construct a carefully chosen η to use in (1).

Let $\rho(r)$ be a C^∞ function on $[0, \infty)$ such that $0 \le \rho(r) \le 1$ and

$$\begin{aligned} \rho(r) &= 1, \quad 0 \le r < \epsilon/2 \\ \rho(r) &= 0, \quad r > \epsilon \end{aligned}$$

and let

$$\omega(r) = -\frac{1}{2\pi}\rho(r)\ln(r)$$

and let

$$\gamma(z, \zeta) = 4\frac{\partial^2}{\partial\bar{z}\partial z}\omega(|z - \zeta|)$$

This is not defined when $z = \zeta$, and we just define it to be 0 for $z = \zeta$. Oviously, $\gamma(z, \zeta) = 0$ when $|z - \zeta| > \epsilon$. Since $\log|z - \zeta|$ is harmonic in z for $z \ne 0$, we also have $\gamma(z, \zeta) = 0$ when $|z - \zeta| < \epsilon/2$.

Now we define the function that we will eventually use as η in (1). Let $\mu(z)$ be C^∞ with support in $D_{1-2\epsilon}$, the disc of radius $1 - 2\epsilon$ centered at 0. Define

$$\psi(z) = \int\int_{\mathbb{D}} \omega(|\zeta - z|)\mu(\zeta)\frac{d\zeta \wedge d\overline{\zeta}}{-2i}$$

The support condition on μ implies we can change the integration region from $\mathbb{D}$ to all of $\mathbb{C}$. Then we can do a change of variables:

$$\begin{aligned} \psi(z) &= \int\int_{\mathbb{C}} \omega(|\zeta - z|)\mu(\zeta)\frac{d\zeta \wedge d\overline{\zeta}}{-2i} \\ &= \int\int_{\mathbb{C}} \omega(|\zeta|)\mu(\zeta + z)\frac{d\zeta \wedge d\overline{\zeta}}{-2i} \end{aligned}$$

Since $\mu(z)$ is C^∞ and compactly supported, standard analysis theorems show ψ is C^∞ in z.

We will take $\eta = \psi$ in (1), so we need to compute the Laplacian of ψ. We split ψ as

$$\psi(z) = \alpha(z) + \beta(z)$$

$$\begin{aligned} \alpha(z) &= \frac{-1}{2\pi}\int\int_{|\zeta - z| < \epsilon/2} \mu(\zeta)\log|\zeta - z|\frac{d\zeta \wedge d\overline{\zeta}}{-2i} \\ \beta(z) &= \int\int_{|\zeta - z| \ge \epsilon/2} \omega(|\zeta - z|)\mu(\zeta)\frac{d\zeta \wedge d\overline{\zeta}}{-2i} \end{aligned}$$

We claim

$$\begin{aligned} 4\frac{\partial^2}{\partial z\partial\overline{z}}\alpha(z) &= -\mu(z) \\ 4\frac{\partial^2}{\partial z\partial\overline{z}}\beta(z) &= \int\int_{\mathbb{D}}\gamma(z,\zeta)\mu(\zeta)\frac{d\zeta\wedge d\overline{\zeta}}{-2i} \end{aligned}$$

Looking at where μ is supported, we see that the integration region in the definition of β can be extended to all of $\mathbb{C}$. Then the equation for the Laplacian of β just follows from the definition of $\gamma(\zeta, z)$.

To compute the Laplacian of α, we start by computing $\partial\alpha/\partial z$. Since $\log|\zeta - z|$ is harmonic in z, its derviative with respect to z is analytic. A simple calculation shows

$$\frac{\partial\log|\zeta - z|}{\partial z} = -\frac{1}{2}\frac{1}{\zeta - z}$$

So we have

$$\frac{\partial\alpha}{\partial z} = \frac{1}{4\pi}\int\int_{|\zeta-z|<\epsilon/2}\frac{\mu(\zeta)}{\zeta - z}\frac{d\zeta\wedge d\overline{\zeta}}{-2i}$$

We would like to change the region of integration to all of $\mathbb{C}$ so we can do a change of variables. But μ is not supported in just the region of integration. So we will "cutoff" μ.

Define $\nu(\zeta) = \rho(2|\zeta - z|)\mu(\zeta)$. Note that $\nu(\zeta)$ is supported in $|\zeta - z| < \epsilon/2$. Now

$$\frac{\partial\alpha}{\partial z}(z) = \frac{1}{4\pi}\int\int_{|\zeta-z|<\epsilon/2}\frac{\nu(\zeta)}{\zeta - z}\frac{d\zeta\wedge d\overline{\zeta}}{-2i} + \frac{1}{4\pi}\int\int_{|\zeta-z|<\epsilon/2}\frac{\mu(\zeta)-\nu(\zeta)}{\zeta - z}\frac{d\zeta\wedge d\overline{\zeta}}{-2i}$$

Now $\mu(\zeta) - \nu(\zeta) = 0$ for $|\zeta - z| < \epsilon/2$. So the last integral is holomorphic in z. So to compute $\partial^2\alpha/\partial z\partial\overline{z}$, we can ignore this term. The support property of ν give

$$\begin{aligned} \frac{1}{4\pi}\int\int_{|\zeta-z|<\epsilon/2}\frac{\nu(\zeta)}{\zeta - z}\frac{d\zeta\wedge d\overline{\zeta}}{-2i} &= \frac{1}{4\pi}\int\int_{\mathbb{C}}\frac{\nu(\zeta)}{\zeta - z}\frac{d\zeta\wedge d\overline{\zeta}}{-2i} \\ &= \frac{1}{4\pi}\int\int_{\mathbb{C}}\frac{\nu(\zeta + z)}{\zeta}\frac{d\zeta\wedge d\overline{\zeta}}{-2i} \end{aligned}$$

So

$$\frac{\partial^2 \alpha}{\partial \overline{z} \partial z}(z) = \frac{1}{4\pi} \int\int_{\mathbb{C}} \frac{\nu_{\overline{z}}(\zeta + z)}{\zeta} \frac{d\zeta \wedge d\overline{\zeta}}{-2i} = \frac{1}{4\pi} \int\int_{\mathbb{C}} \frac{\nu_{\overline{z}}(\zeta)}{\zeta - z} \frac{d\zeta \wedge d\overline{\zeta}}{-2i} \tag{2}$$

To continue the derviation of the formula for the Laplacian of α, we need the following "Cauchy integral formula".

Lemma 1. (Cauchy integral formula for non-analytic functions) *Let B be an open connected subset of $\mathbb{C}$ whose boundary is a finite number of C^1 Jordan curves. Let u be a complex valued C^1 function on B which is not necessarily analytic. Then for $z \in B$,*

$$2\pi i u(z) = \int_{\partial B} \frac{u(\zeta)}{\zeta - z} d\zeta + \int\int_B \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

Note that in (2) we can replace the integration region by a region that we can apply the lemma to. This yields

$$\frac{\partial^2 \alpha}{\partial \overline{z} \partial z}(z) = -\frac{1}{4}\mu(z)$$

We now use (1) with $\eta = \psi$. We get

$$\begin{aligned} 0 \quad &= \quad \int\int_{\mathbb{D}} \phi \Delta \psi = \int\int_{\mathbb{D}} \phi \Delta \alpha + \int\int_{\mathbb{D}} \phi \Delta \beta = -\int\int_{\mathbb{D}} \phi(z)\mu(z) \frac{dz \wedge d\overline{z}}{-2i} \\ &+ \quad \int\int_{\mathbb{D}} \phi(z) \left[\int\int_{\mathbb{C}} \gamma(z,\zeta)\mu(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{-2i} \right] \frac{dz \wedge d\overline{z}}{-2i} \end{aligned}$$

Define

$$\tilde{\phi}(\zeta) = \int\int_{\mathbb{C}} \phi(z)\gamma(z,\zeta) \frac{dz \wedge d\overline{z}}{-2i}$$

Note that $\tilde{\phi}$ is C^∞. We now have

$$\int\int_{\mathbb{D}} \phi(z)\mu(z) \frac{dz \wedge d\overline{z}}{-2i} = \int\int_{\mathbb{D}} \tilde{\phi}(\zeta)\mu(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{-2i}$$

This holds for all C^∞ functions that are compactly supported in $\mathbb{D}$. So $\phi = \tilde{\phi}$ a.e. on $\mathbb{D}$.

Proof. **(of Cauchy integral formula)** Fix a z in B and let $\epsilon > 0$ be small enough that the ball of radius ϵ about z is contained in B. Let B_ϵ be B minus the closed ball of radius ϵ about z. The boundary of B_ϵ is the boundary of B plus the circle of radius ϵ about z. Stoke's theorem says

$$\int_{\partial B_\epsilon} \frac{u(\zeta)}{\zeta - z} d\zeta = \int \int_{B_\epsilon} d \left[\frac{u(\zeta)}{\zeta - z} \right] d\zeta$$

which becomes

$$\int_{\partial B} \frac{u(\zeta)}{\zeta - z} d\zeta - i \int_0^{2\pi} u(z + \epsilon e^{i\theta}) d\theta = \int \int_{B_\epsilon} \frac{\partial}{\partial \overline{\zeta}} \left[\frac{u(\zeta)}{\zeta - z} \right] d\overline{\zeta} \wedge d\zeta$$

Let $\epsilon \to 0$ to get the lemma. $\square$

2.3 The Hilbert space of forms

Let D be a region in M. Recall that the inner product for measurable 1-forms for $L^2(D)$ is given by

$$(\omega_1, \omega_2) = \int \int_D \omega_1 \wedge * \overline{\omega_2}$$

If D is covered by a single chart and $\omega = u dz + v d\overline{z}$, we have

$$(\omega_1, \omega_2) = 2 \int \int_D (|u|^2 + |v|^2) dx dy$$

Definition 1. *E is the closure in $L^2(M)$ of*

$$\{df : f \, is \, C^\infty \, function \, on \, M \, with \, compact \, support\}$$

And E^* is

$$E^* = \{\omega \in L^2(M) : *\omega \in E\} = \{*\omega \in L^2(M) : \omega \in E\}$$

The second characterization of E^* follows from $** = -1$ and the fact that E is a subspace.

A remark is in order. If M is compact, then every function on M is compactly supported. So in this case E is the space of smooth exact forms. However, in general E does not contain all the smooth exact forms. Suppose

M is the unit disc. Let f be a smooth harmonic function on an open set that contains the closed unit disc. Then df is an exact 1-form. It is harmonic, and we will see later that this implies it is not in E.

By definition $E^\perp$ is the set of ω orthogonal to all the forms in E. By continuity of the inner product it suffices to only require that they be orthogonal to a dense subset of E. So

$$E^\perp = \{\omega : (\omega, df) = 0 \quad \forall \quad f\, which\, are\, C^\infty\, function\, on\, M\, with\, compact\, support\}$$

Similarly,

$$(E^*)^\perp = \{\omega : (*\omega, df) = 0 \quad \forall f \cdots\}$$

Proposition 1. *Let α be a C^1 form. Then $\alpha \in (E^*)^\perp$ if and only if α is closed. And $\alpha \in E^\perp$ if and only if α is co-closed.*

Proof. Let α be a C^1 form and f a C^1 function with compact support. So there is an open set D containing the support of f such that the closure of D is compact. Then

$$(\alpha, *df) \;=\; \int_D \alpha \wedge * * d\overline{f} = -\int_D \alpha \wedge d\overline{f} = -\int_D [d(\alpha\overline{f}) - d\alpha \wedge \overline{f}]$$

By Stokes theorem,

$$\int_D d(\alpha\overline{f}) = \int_{\partial D} \alpha\overline{f} = 0$$

by the support assumption. So

$$(\alpha, *df) = \int_D d\alpha \wedge \overline{f}$$

If α is closed, then this gives $(\alpha, *df) = 0$. So $\alpha \in (E^*)^\perp$. If $\alpha \in (E^*)^\perp$, we have $(\alpha, *df) = 0$. So

$$\int_D d\alpha \wedge \overline{f} = 0$$

for all C^∞ f with compact support. This implies $d\alpha = 0$, i.e., α is closed.

The second part follows immediately since E^* is just the image of E under the $*$ map and the co-closed forms are the image of the closed forms under the map $*$. □

Corollary 1. *E and E^* are orthogonal subspaces.*

Proof. Let $\omega \in E$. Then there is a sequence of C^∞ functions such that df_n converges to ω in L^2. Since df_n is exact, it is closed. So by the proposition $df_n \in (E^*)^\perp$. Since $(E^*)^\perp$ is closed, ω belongs to it as well. ☐

Now we define $H = E^\perp \cap (E^*)^\perp$. ($H$ stands for harmonic; we will see that the forms in H are harmonic.) Since E and E^* are orthogonal, Hilbert space theory says

$$\begin{aligned} H &= (E \oplus E^*)^\perp \\ L^2(M) &= E \oplus E^* \oplus H \end{aligned}$$

Our next goal is to show that H consists exactly of the harmonic forms.

Let c be a simple closed curve in our Riemann surface M. Cover c by a finite number of charts. Let Ω be the union of their domains. Shrinking Ω if necessarily we can assume that topologically it is an annulus and that $\Omega \setminus c$ is two strips. The curve has a direction. Let Ω^- (Ω^+)be the strip in $\Omega \setminus c$ that is on the "left" ("right") as we traverse c. Let Ω_0 be a subset of Ω which is smaller in the sense that the boundary of Ω_0 has no points in common with that of Ω. Define $\Omega_0^\pm$ in the same way.

Let f be a real-valued function on M with is 1 on Ω_0^- and 0 on $M \setminus \Omega^-$ and such that f is smooth on $M \setminus c$. We next define a smooth 1-form η by

$$\begin{aligned} \eta_C &= df, \quad \mathit{on} \quad \Omega \setminus c \\ \eta_C &= 0, \quad \mathit{on} \quad (M \setminus \Omega) \cup c \end{aligned}$$

Then η is a closed, smooth, compactly supported real value 1-form. It is not in general exact. We call η_c the 1-form associated with the curve c. (It is not unique).

Proposition 2. *Let $\alpha \in L^2(M)$ be C^1 and closed. Then*

$$\int_c \alpha = (\alpha, *\eta_c)$$

Proof. Homework problem. ☐

Proposition 3. *Let $\alpha \in L^2(M)$ be C^1. Then α is exact (respectively co-exact) if and only if $(\alpha, \beta) = 0$ for all co-closed (closed) smooth differentials β of compact support.*

Proof. Let α be C^1 and exact, and let β be co-closed with compact support. So the support of β is contained in some region D whose closure is compact. We have

$$(\alpha, \beta) = \int_D df \wedge *\overline{\beta} = \int_D [d(f * \overline{\beta}) - fd * \overline{\beta}] = \int_{\partial D} f * \overline{\beta} = 0$$

To go the other way, suppose $(\alpha, \beta) = 0$ for all co-closed smooth compactly supported β. The closure of the set of all such β is E^*, so $\alpha \in (E^*)^{\perp}$. By prop 1 this implies α is closed. This implies it is locally exact. For global exactness we need to show that $\int_c \alpha = 0$ for all simple closed curves. This follows from prop 2 and the $(\alpha, \beta) = 0$ hypothesis. $\square$

Theorem 2. *H is the set of harmonic differentials in $L^2(M)$.*

Proof. By a proposition from chapter 1, if ω is harmonic then it is smooth, closed and co-closed. By proposition 1, this shows it is in both $E^{\perp}$ and $(E^*)^{\perp}$. So $\omega \in H$.

Now let $\omega \in H$. Harmonicity is a local property, so it suffice to show ω in harmonic on a chart (D, z). We can assume the closure of D is compact. Write $\omega = pdx + qdy$. Let η be a smooth real-valued function supported in D. Then η_x and η_y are also smooth and supported in D. (The subscripts means partial with respect to that variable.) So $\eta_x \in E$ and $*\eta_y \in E^*$. Since ω is orthogonal to both of these subspaces,

$$\begin{aligned} 0 &= (\omega, d\eta_x) = \int\int_D \omega \wedge *d\eta_x = \int\int_D (pdx + qdy) \wedge *(\eta_{xx}dx + \eta_{xy}dy) \\ &= \int\int_D (pdx + qdy) \wedge (-\eta_{xy}dx + \eta_{xx}dy) = \int\int_D (p\eta_{xx} + q\eta_{xy})dx \wedge dy \end{aligned}$$

$$\begin{aligned} 0 &= (\omega, *d\eta_y) = \int\int_D \omega \wedge d\eta_y = \int\int_D (pdx + qdy) \wedge (\eta_{yx}dx + \eta_{yy}dy) \\ &= \int\int_D (p\eta_{yy} - q\eta_{yx})dx \wedge dy \end{aligned}$$

Adding these equations gives

$$0 = \int \int_D p(\eta_{xx} + \eta_{yy}) dx \wedge dy$$

By Weyl's lemma this implies that p is smooth and harmonic. Since $*\omega \in H$, we can apply the same argument to $*\omega$ which shows q is smooth and harmonic. All we really need from this is that p and q are C^1, and so ω is C^1. So by proposition 1, we can conclude ω is closed and co-closed. By the proposition from chapter 1 this implies it is harmonic. □

Corollary 2. *(a) The closure (in $L^2(M)$) of the space of closed forms is $E \oplus H$. The closure of the space of co-closed forms is $E^* \oplus H$.*
(b) The square integrable smooth differentials are dense in $L^2(M)$.
(b') The smooth differentials with compact support are dense in $L^2(M)$.

Proof. The proofs of (a) and (b) follow from putting together various things we have proved and left as a homework exercise. The proof of (b') requires knowning a topolocial fact about M that we haven't proved yet. See the book for details. □

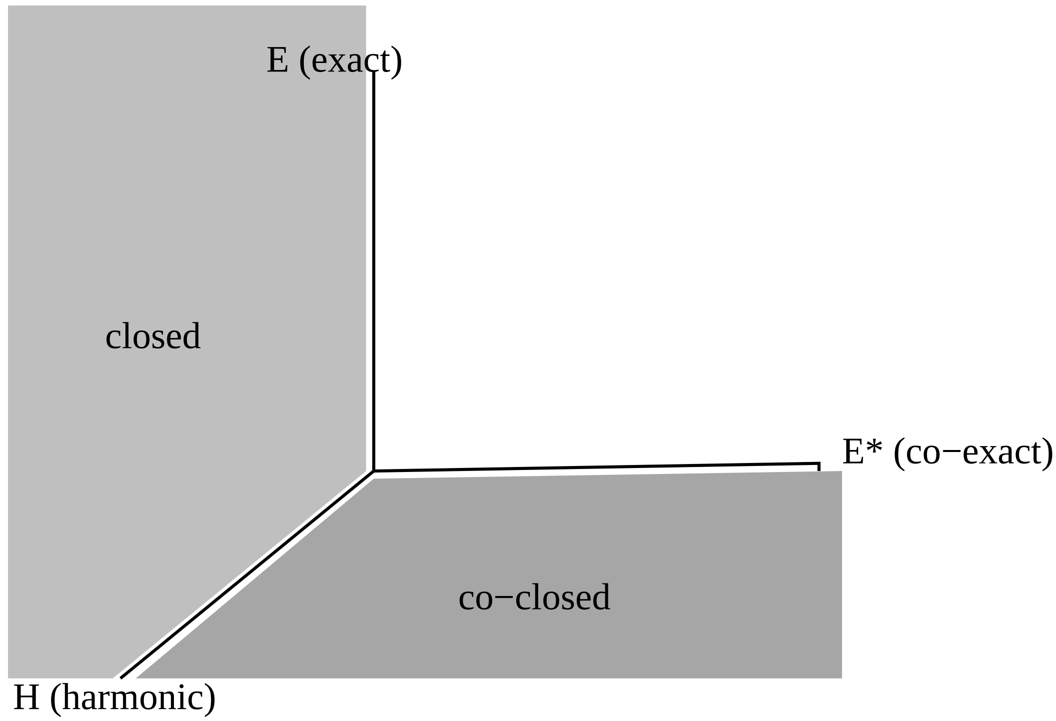
We repeat a remark we made before: if M is not compact, then not all exact forms are in E.

2.4 The Hilbert space of forms

Recall that a 1-form is harmonic if it is locally given by df where f is a harmonic function. Harmonic diferentials immediately give us holomorphic 1-forms. (Recall that if α is harmonic, then $\alpha + i * \alpha$ is holomorphic.)

On a compact Riemann surface there are no holomorphic functions (other than constants). We want to prove there do exist meromorphic functions and to do that we first construct meromorphic differentials. These will follow immediately from the construction of differentials which are harmonic except at a point. To do this we want functions that are harmonic except at a point.

Consider for a moment the Riemann surface $\mathbb{C}$. There are harmonic functions, but not ones in L^2. Suppose we allow a singularity at the origin. Then z^{-n} is an example of a harmonic on $\mathbb{C} \setminus \{0\}$. It is not L^2 but it is if we chop out a small neighborhood of the origin. Our goal in this section is to show that for a Riemann surface M and a point P_0 in it, we can find a function which is harmonic on $M \setminus \{P_0\}$ and whose singularity at P_0 is (in local coords) z^{-n}.



Theorem 3. *Let M be a Riemann surfact, $P_0 \in M$. Let z be a local coordinate centered at P_0. Then there is a function u on $M \setminus \{P_0\}$ such that $u - z^{-n}$ is harmonic on a neighborhood of P_0. For any neighborhood N of P_0,*

$$\int\int_{M \setminus N} du \wedge *\overline{du} < \infty$$

Furthermore $(du, df) = 0 = (du, *df)$ for all smooth functions that vanish in a neighborhood of P_0 and have compact support.

Note that if M is compact and u_1, u_2 are two such functions, then their difference is harmonic on all of M. So it is a constant. Thus for compact M the function in the theorem is unique up to an additive constant.

Proof. Let D be the domain of our chart z. We can assume its range is the unit disc. (A subdomain of D is mapped onto some disc centered at the origin, and then we rescale.) Let $0 < a < 1$. Define

$$\begin{aligned} h(z) &= z^{-n} + \frac{\overline{z}^n}{a^{2n}}, \quad if \quad |z| \le a \\ h(z) &= 0 \quad otherwise \end{aligned}$$

$$\begin{aligned} \theta(z) &= h(z) \quad if \quad |z| \ge a/2 \\ \theta(z) &= smooth \quad if \quad |z| < a \end{aligned}$$

We use $d\theta$ to denote the form which is $d\theta$ where θ is smooth and is whatever you want on $|z| = a$ where θ is not smooth. This is in $L^2(M)$. Note that $d\theta$ is not exact. We use the theory from the last section to split it as

$$d\theta = \alpha + \omega$$

where $\alpha \in E$ and $\omega \in E^\perp = E^* \oplus H$. We will show two things: (1) α is smooth on M. (2) α is harmonic on $M \setminus Cl(D_{a/2})$. We first assume these are true and finish the proof.

Since α is smooth, it is exact if and only if it is orthogonal to all co-closed smooth 1-forms with compact support. Such forms are in $E^* \oplus H$, and α is in E, so it is orthogonal. So α is exact, i.e., $\alpha = df$ for some smooth function f on M.

We claim that this f is harmonic on $M \setminus Cl(D_{a/2})$. Since α is harmonic on this set, we can write locally $\alpha = dh$ for some locally harmonic function h. On this neighborhood, $dh = df$, so $\Delta f = \Delta h = 0$ on the nbhd. So f is harmonic on $M \setminus ClD_{a/2}$.

By the original splitting, $\omega = d\theta - \alpha = d(\theta - f)$. So $d(\theta - f) \in E^\perp$. θ is smooth on D_a and f is smooth everywhere, so $d(\theta - f)$ is co-closed on D_a. On D_a, $d(\theta - f)$ is exact and so is closed on D_a, so $d(\theta - f)$ is harmonic on D_a and so $\theta - f$ is harmonic on D_a.

Now define

$$u = f - \theta + h$$

On D_a, $f - \theta$ and h are harmonic, so u is too. On $|z| \ge a/2$, f is harmonic and $h - \theta = 0$. So u is harmonic on $M \setminus \{P_0\}$.

$\square$

We consider just what properties of h were needed for the proof. There really just two: h is harmonic in $a/2 < |z| < a$ and we needed that $*d\theta$ is zero in the direction of the circle $|z| = a$.

As before, let P_0 be a point in the Riemann surface and z a chart centered at P_0. Suppose there are two points P_1, P_2 that are close enough to P_0 that their coordinates $w_1 = z(P_1)$ and $w_2 = z(P_2)$ satisfy $|w_1|, |w_2| < a/2$. Then

$$h(z) = \log \left| \frac{(z - w_1)(z - a^2/\overline{w_1})}{(z - w_2)(z - a^2/\overline{w_2})} \right|$$

Theorem 4. *Let M be a Riemann surfact, $P_1, P_2 \in M$. Let z_i be a local coordinates centerd at P_i for $i = 1, 2$. Then there is a real valued function u on which is harmonic on $M \setminus \{P_1, P_2\}$ and such that $u - log|z_1|$ is harmonic in a nbhd of P_1 and $u + log|z_2|$ is harmonic in a nbhd of P_2. For every open set N containing P_1 and P_2,*

$$\int_{M \setminus N} du \wedge *\overline{du} < \infty$$

Furthermore $(du, df) = 0 = (du, *df)$ for all smooth functions that vanish in neighborhoods of P_1 and P_2 and have compact support.

Proof. We find points $P_1 = Q_0, Q_1, \cdots, Q_n = P_2$ such that pairs $Q_{j=-1}, Q_j$ are close enough that we can apply the previous theorem. Let u_j be the harmonic (with singularities) function. Then we just take $u = u_1 + \cdots + u_n$. □

3 Meromorphic functions and differentials

Definition 2. *A meromorphic differential ω is an assignment of a meromorphic function $f(z)$ to every chart z so that $f(z)dz$ tranform appropriately under a coordinate change, i.e., they define a 1-form. We say ω has a pole at a point in the domain of the chart if $f(z)$ does. Suppose ω has a pole at p. Let z be a chart centered at p. Writing $\omega = f(z)dz$, $f(z)$ has a Laurent series:*

$$f(z) = \sum_{n=N}^{\infty} a_n z^n$$

where N is negative. We define the residue of the form at p to be a_{-1}.

One of the homework problems is to show that the definition of residue is well defined, i.e., does not depend on the chart.

Let ω be meromorphic and $\omega = f(z)dz$ in some chart. If $f(z)$ has a zero at P, we define the order of f at P to be the order of the zero of $f(z)$. If $f(z)$ has a pole at P we define the order to be $-n$ where n is the order of the pole. We denote the order in both cases by $ord_P f$.

Theorem 5. *(a) Let $P \in M$ and let z be a coordinate centered at P. Then for each positive integer n, there is a meromorphic differential ω on M whose only pole is at P and whose singular part there is z^{-n-1}.*
(b) Let P_1, P_2 be distinct points in M. Let z_j be coordinates centered at P_j for $j = 1, 2$. There is a meromorphic differential ω whose only poles are at P_1, P_2. Its singular parts there are $1/z_1$ and $-1/z_2$, respectively.

Proof. For (a) let u be the function with is harmonic except to a singularity of $1/z^n$ at P. For (b) let u be the function with is harmonic except for singularity $\log(z_1)$ at P_1 and $-\log(z_2)$. Let $\alpha = du$. So α is a harmonic form except at ... We define

$$\begin{aligned} \omega &= \frac{-1}{2n}(\alpha + i * \alpha) \\ \omega &= \alpha + i * \alpha \end{aligned}$$

□

Theorem 6. *Let $k > 1$ and let $P_1, \cdots, P_k$ be distinct points on a Riemann surface M. Let $c_1, \cdots, c_k \in \mathbb{C}$ such that $\sum_{j=1}^k c_j = 0$. Then there is a meromorphic differential whose poles are at $P_1, \cdots, P_k$ such that $ord_{P_j} \omega = -1$ and $res_{P_j} \omega = c_j$ for $j = 1, \cdots, k$.*

Proof. Let P_0 be a point in M distinct from all the P_j. Let ω_j be a meromorphic differential with poles at P_0 and P_j. Both poles have order -1. The residue at P_j is $+1$ and the residue at P_0 is -1. Then we just let

$$\omega = \sum_{j=1}^{k} c_j \omega_j$$

The hypothesis that $\sum_j c_j = 0$ implies that the residue at P_0 is zero, i.e., there is no pole there. □

Corollary 3. *Every Riemann surface has non-constant meromorphic functions*

Proof. Let P_1, P_2, P_3 be distinct points in the surface. Let ω_1, ω_2 be meromorphic differentials such that ω_1 has poles at P_1, P_2 and ω_2 has poles at P_2, P_3 and

$$ord_{P_1}\,\omega_1 = ord_{P_2}\,\omega_1 = ord_{P_2}\,\omega_2 = ord_{P_3}\,\omega_2 = -1$$

and

$$\begin{gathered} res_{P_1}\,\omega_1 = +1, res_{P_2}\,\omega_1 = -1, \\ res_{P_2}\,\omega_2 = +1, res_{P_3}\,\omega_1 = -1 \end{gathered}$$

Define $f = \omega_1/\omega_2$. (Just what this means will be the subject of a homework problem.) Then f is meromorphic with a pole at P_1 and a zero at P_3. In particular it is not constant. □

Proposition 4. *Let M be a compact Riemann surface and ω a meromorphic differential. Then*

$$\sum_{P \in M} res_P\,\omega = 0$$

The sum is over the poles and zeroes of ω.

We defer the proof till later.