Math 520b - Homework 1

1. In this problem we prove that a couple of results for ordinary analytic function on domains in $\mathbb{C}$ have analogs for maps between Riemann surfaces.
(a) Let f be an analytic map between Riemann surfaces, $f : M \to N$. Let $q \in N$ and suppose there exist a sequence $p_n \in M$ which converges to some $p \in M$ and for which $f(p_n) = q$. In other words, the set $f^{-1}(q)$ has an accumulation point. Prove that f is constant.
(b) Let f be an analytic map between Riemann surfaces which is not constant. Prove that f is an open map, i.e., it maps open sets to open sets.

2. This exercise is about the Riemann sphere example.
(a) Compute the transition function for the two charts for the Riemann sphere and check that it is analytic.
(b) Consider a different possible sets of charts. Let

$$\begin{aligned}
U_1^+ &= \{(x, y, w) : x > 0\}, \quad \phi_1^+(x, y, w) = y + iw \\
U_1^- &= \{(x, y, w) : x < 0\}, \quad \phi_1^-(x, y, w) = y + iw \\
U_2^+ &= \{(x, y, w) : y > 0\}, \quad \phi_2^+(x, y, w) = x + iw \\
U_2^- &= \{(x, y, w) : y < 0\}, \quad \phi_2^-(x, y, w) = x + iw \\
U_3^+ &= \{(x, y, w) : w > 0\}, \quad \phi_3^+(x, y, w) = x + iy \\
U_3^- &= \{(x, y, w) : w < 0\}, \quad \phi_3^-(x, y, w) = x + iy
\end{aligned}$$

These are legitimate charts if we want to treat the sphere as a real two dimensional smooth manifold. Are they legitimate complex charts? You should of course explain your answer.

3. This exercise is about the projective line example.
(a) Compute the transition function for the two charts for the complex projective line and check that it is analytic.
(b) Show that the following map gives an isomorphism from $\mathbb{CP}^1$ to $\hat{\mathbb{C}}$:

$$[z : w] \to \frac{(2Re(z\overline{w}), 2Im(z\overline{w}), |z|^2 - |w|^2)}{|z|^2 + |w|^2}$$

4. This exercise verifies some facts that are only true because we are on a Riemann surface, not just a smooth real two dimensional manifold.
(a) $*\omega$ is well defined and is a 1-form.

(b) Recall that the Laplacian Δ takes a C^2 function on the Riemann surface to a 2-form. Show that it is well defined, i.e., independent of the choice of chart.

5. In class I said that a meromorphic function on a Riemann surface was a holomorphic function to $\hat{\mathbb{C}}$ that is not identically ∞. This problem fills in the details behind that remark. Forget the preceding definition of meromorphic. Recall that a function on a domain in $\mathbb{C}$ is meromorphic if its singularities are isolated and they are only poles. Now let f be a complex valued function on M minus a set of points with no accumulation point. We define f to be meromorphic if for every chart $\{U, \phi\}$, $f \circ \phi^{-1}$ is meromorphic on U. For such a function define $F(p)$ to be $f(p)$ if p is not a pole and ∞ if p is a pole. So F maps M to $\hat{\mathbb{C}}$. Prove that this construction give a 1-1 correspondence between meromorphic functions f on M and holomorphic functions $F : M \to \hat{\mathbb{C}}$ which are not identically ∞.