Math 520b - Homework 2

1. Let H be a Hilbert space. Let E_1, E_2 be closed subspaces which are orthogonal. Prove there is a third closed subspace E_3 which is orthogonal to both E_1 and E_2 and such that every $v \in H$ can be written in a unique way as $v = v_1 + v_2 + v_3$ where $v_i \in E_i$. Show that $E_3 = (E_1 \oplus E_2)^\perp = E_1^\perp \cap E_2^\perp$. You may assume the projection theorem as stated in the notes or in Farkas and Kra.

2. (From Farkas and Kra) (a) Let $f \in L^2[0,1]$. Show that f is equal to a constant almost everywhere if and only if

$$\int_0^1 f(z) g'(x) dx = 0 \tag{1}$$

for all C^∞ functions with compact support. You can think of this as a 1d analog of Weyl's lemma.
(b) If we place the hypothesis (1) by

$$\int_0^1 f(z) g''(x) dx = 0 \tag{2}$$

what can you conclude about f ?

3. Let ϕ be a C^2 function on the unit disc that is harmonic. Prove that ϕ is C^∞.

4. Prove parts (a) and (b) of the corollary to theorem 2 which says that H is the set of harmonic forms.

5. A meromorphic differential (or 1-form) ω is a 1-form such that in every chart z, $\omega = f(z)dz$ where $f(z)$ is meromorphic. The form is said to have a pole at p if f does. Suppose ω has a pole at p. Let z be a chart centered at p. Writing $\omega = f(z)dz$, $f(z)$ has a Laurent series:

$$f(z) = \sum_{n=N}^{\infty} a_n z^n$$

where N is negative. We define the residue of the form at p to be a_{-1}. In general the coefficients in the Laurent series depend on the choice of chart. Prove that the residue does not depend on the choice of chart. Hint: how do you compute the coefs in a Laurent series?