Math 520b - Homework 3

1. Let ω_1, ω_2 be two meromorphic 1-forms. Let z be a chart. On its domain we define ω_1/ω_2 to be $f_1(z)/f_2(z)$ where $\omega = f_i(z)dz$. Show that this definition is well defined, i.e., it doesn't depend on the choice of chart. So this defines a meromorphic function.

2. Prove the following complex varaibles proposition:

Proposition Let f be analytic in a neighborhood of 0. Let $w_n \in \mathbb{C}$ with $w_n \to 0$. For each n, let $z_n^1, \cdots, z_n^m$ be distinct complex numbers in the domain of f such that $f(z_n^j) = w_n$ for $j = 1, \cdots, m$. Suppose also that for $j = 1, \cdots, m$, $\lim_{n\to\infty} z_n^j = 0$. Then 0 is a zero of f with multiplicity at least m.

3. Recall that for a meromorphic differential on a compact Riemann surface, the sum of the residues at the poles is zero. Consider the Riemann sphere. Let z be the chart on $\hat{\mathbb{C}} \setminus \{\infty\}$ which comes from stereographic projection. Consider the form $\omega = dz/z$. Clearly it has a pole at $z = 0$, the south pole, with residue 1. As $z \to \infty$, $1/z \to 0$, so there is no pole at $z = \infty$, the north pole. So the sum of the residues is 1. What is wrong with this "counterexample"?

4. Let f be a meromorphic function on a compact Riemann surface that is not a constant. Recall that if f has a zero of order n at P then we define $ord_P(f) = n$. If f has a pole of order n at P then we define $ord_P(f) = -n$. Prove that $\sum_P ord_P(f) = 0$. Hint: a meromorphic function is a holomorphic map of M to $\hat{\mathbb{C}}$.

5. Consider the Riemann sphere $\hat{\mathbb{C}}$. Let z be the chart which comes from stereographic projection. Define a meromorphic function f on $\hat{\mathbb{C}}$ by

$$f(z) = \frac{z^3}{1 - z^2}$$

and $f(\infty) = \infty$. (∞ is the north pole of the sphere.) We can think of this as a homomorphic map of $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Find the degree of this map and find its ramification points and their orders.

6. Let M be a compact Riemann surface. We know that it has a non-constant meromorphic function f on it. The goal of this problem is to use f to prove

that M has a triangulation. The rough idea is to find a triangulation of $\hat{\mathbb{C}}$ that lifts to M. Since M is compact, it only as finite number of points with ramification number greater than one. Suppose $U \subset \hat{\mathbb{C}}$ is open such that f^{-1} does not contain any ramified points. Let n be the degree of f. Then $f^{-1}(U)$ is the disjoint union of n open sets which are homeomorphic to each other. A triangulation of U lifts to triangulations on these open sets. Show that if we take a triangulation of $\hat{\mathbb{C}}$ such that every point with ramification number greater than one gets mapped by f to a vertex of the triangulation and the triangles are "small enough," then this triangulation lifts to a triangulation of M.

7. We sketched the proof of

Theorem 1. *Let M be a Riemann surfact, $P_1, P_2 \in M$. Let z_i be a local coordinates centerd at P_i for $i = 1, 2$. Then there is a real valued function u which is harmonic on $M \setminus \{P_1, P_2\}$ and such that $u - log|z_1|$ is harmonic in a nbhd of P_1 and $u + log|z_2|$ is harmonic in a nbhd of P_2. ...*

We used the harmonic function

$$h(z) = \log \left| \frac{(z - w_1)(z - a^2/\overline{w_1})}{(z - w_2)(z - a^2/\overline{w_2})} \right|$$

where $|w_1|, |w_2| < a/2$. Show that $*dh = 0$ "along" $|z| = a$.