Math 520b - Homework 4

1. Let M be a compact Riemann surface of genus g. Let $P_1, \cdots, P_k \in M$ and let $n_1, \cdots, n_k$ be positive integers. Find the dimension of the vector space of meromorphic forms whose poles are a subset of $\{P_1, \cdots, P_k\}$ and such that the order of the pole at P_j (if there is one) is at most n_j.

2. Prove that for the Riemann sphere two divisors U_1 and U_2 are equivalent if and only if $deg(U_1) = deg(U_2)$.

3. Let $\tau \in \mathbb{C}$ have positive imaginary part. Let L be the lattice $\mathbb{Z} + \mathbb{Z}\tau$. Let M be the Riemann surface which is the torus $\mathbb{C}/L$. The meromorphic functions on M are the meromorphic functions on $\mathbb{C}$ which are doubly periodic with periods 1 and τ. For $p \in \mathbb{C}$ let γ_p be the closed contour which is the parallelogram with vertices $p, p+1, p+1+\tau, p+\tau$ traversed in that order. Let f be a doubly periodic meromorphic function on $\mathbb{C}$. Show that for p such that γ_p does not contain any poles or zeroes of f, we have

$$\frac{1}{2\pi i} \int_{\gamma_p} z \frac{h'(z)}{h(z)} dz \in L$$

4. Let M be the torus defined in the previous problem. Consider the divisor $U = P_1^{n_1} \cdots P_k^{n_k}$. We know that if it is principal, then $deg(U) = 0$. Define

$$A(U) = \sum_{j=1}^{k} n_j P_j$$

(Addition of points in the torus M is addition in $\mathbb{C}$ mod L.) Prove that if U is principal, then $A(U) = 0$. (We think of $A(U)$ as an element of the torus, so $A(U) = 0$ means it is 0 mod L.) This is "half" of Abel's theorem for the torus which says that a divisor U is principal if and only if $deg(U) = 0$ and $A(U) = 0$.

5. Let M be a compact Riemann surface. Let $P_1, P_2, \cdots, P_n$ be distinct points in M. Prove that there is a meromorphic form such that for all j the form is non-zero at P_j and does not have a pole at P_j. Hint: Let Q be a point distinct from all the P_j. For large N what is the dimension of $L(Q^{-N})$ and of $L(Q^{-N} P_j)$?