Math 520b - Homework 5

1. In class we used but did not prove the following proposition.
Proposition: Let A be finite dimensional space of holomorphic functions on a domain $D \subset \mathbb{C}$. Let $\phi_1, \phi_2, \cdots, \phi_n$ be a basis. Define

$$\Phi(z) = det \begin{pmatrix} \phi_1(z) & \phi_2(z) & \cdots & \phi_n(z) \\ \phi_1'(z) & \phi_2'(z) & \cdots & \phi_n'(z) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1^{(n-1)}(z) & \phi_2^{(n-1)}(z) & \cdots & \phi_n^{(n-1)}(z) \end{pmatrix}$$

Then the order of the zero of ϕ at z is $\tau(z)$.

In this problem we prove it. We abbreviate the above determinant by $[\phi_1(z), \phi_2(z), \cdots, \phi_n(z)]$.

(a) Let $f(z)$ be a holomorphic function on D. Use properties of determinants to show

$$det[f\phi_1, f\phi_2, \cdots, f\phi_n] = f^n det[\phi_1, \phi_2, \cdots, \phi_n]$$

(b) Prove the prop by induction. Hint

$$det[\phi_1, \phi_2, \cdots, \phi_n] = \phi_1^n \, det[1, \phi_2/\phi_1, \cdots, \phi_n/\phi_1]$$

2. Let f be meromorphic function on a Riemann surface M. Let c be a closed curve in M. Show that

$$\frac{1}{2\pi i} \int_c \frac{df}{f}$$

is an integer.

3. Let ω_1, ω_2 be nonzero complex numbers such that their ratio is not real. Let L be the lattice they generate:

$$L = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$$

Then $\mathbb{C}/L$ is a Riemann surface (the torus). Prove that two tori are the same Riemann surface (conformally isomorphic) if and only if they have the same ω_1/ω_2.

4. In this problem we use Abel's thm to prove that the only compact Riemann surfaces of genus 1 are the tori. Let M be a Riemann surface with genus 1. Then the Jacobian variety $J(M)$ is a torus. (You should check this is true but need not write anything.) We will show $\phi : M \to J(M)$ is a holomorphic bijection. The first two parts are easy. The third part is the interesting one.
(a) Prove ϕ is holomorphic.
(b) Prove ϕ is surjective.
(c) Prove ϕ is injective. Hint: suppose there are distinct points P, Q in M with $\phi(P) = \phi(Q)$ in $J(M)$. Use Abel's theorem to show there is a meromorphic function in $L(D)$ where D is the divisor P/Q. Show this contradicts what we know about gaps.

5. Let P, Q be distinct points in a compact Riemann surface M. Let τ_{PQ} be a meromorphic differential form with simple poles at P and Q, no other poles, residue $+1$ at P and -1 at Q. Let a_j, b_j be a canonical homology basis $(j = 1, 2, \cdots, g)$, and ζ_j the usual dual basis for holomorphic forms. Suppose also that

$$\int_{a_j} \tau_{PQ} = 0, \quad j = 1, 2, \cdots g$$

Show that

$$\int_{b_j} \tau_{PQ} = 2\pi i \int_Q^P \zeta_j$$

where the curve from Q to P does not cross any of the a, b curves.
Hint: Mimic what we did in class to compute $\int_{b_j} \tau_P^{(n)}$ where $\tau_P^{(n)}$ is the differential form with singular part dz/z^n at P, no other poles and $\int_{a_j} \tau_P^{(n)} = 0$.