Math 520b - Homework 6

1. (Mirandi, p. 112 G) Let $h(x)$ be a polynomial of degree $2g+1$ or $2g+2$ with distince roots. Let M be the hyperelliptic Riemann surface given by $y^2 = h(x)$. Show that dx/y is a holomorphic 1-form on M. More generally, show that if $p(x)$ is a polynomial of degree at most $g-1$, then $p(x)dx/y$ is a holomorphic 1-form. Prove that every holomorphic 1-form is of this form.

2. (Mirandi, p. 137 A) Let M be the hyperelliptic surface given by $y^2 = x^5 - x$. Find the principal divisors (x) and (y).

3. (Mirandi, p. 84 H) Let $\hat{\mathbb{C}}$ be the Riemann sphere and let z be the chart which comes from stereographic projection with respect to the north pole. Let R_n be the conformal automorphism $R_n(z) = \exp(2\pi i/n)z$ (rotation by $2\pi/n$). Let F be the conformal automorphism $F(z) = 1/z$. Let H be the group generated by R_n and F. What is this group? Find the branch points and their ramification number for the quotient map $\pi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}/H$.

4. Let H be a finite group of conformal automorphisms of a Riemann surface. Recall that for $P \in M$, H_P is the stabilizer subgroup of P, i.e., the set of $h \in H$ such that $h(P) = P$. Show that the points P for which H_P is not trivial are isolated, i.e., if P is such a point then there is a neighborhood of P in which P is the only point with a nontrivial stabilizer subgroup.

5. Prove Harnack's inequality which I stated in class:

Let D be a domain in $\mathbb{C}$, D_1 a bounded subdomain of D such that $\delta = dist(D_1, \partial D) > 0$. Then there is a constant $c = c(D_1, D)$ such that for all positive harmonic functions u,

$$\frac{1}{c} \le \frac{u(z_1)}{u(z_2)} \le c, \quad \forall z_1, z_2 \in D_1$$

Note that the constant c does not depend on u.