Math 520b - Homework 7

1. (Farkas and Kra, p. 187) Show that the Green's function is a conformal invariant, i.e., if $f : M \to N$ is conformal and g is the Green's function on N with singularity at $f(P)$, then $g \circ f$ is the Green's function on M with singularity at P.

2. (a) Show that a simply connected subset of $\mathbb{C}$ which is not all of $\mathbb{C}$ is hyperbolic
(b) Show that $\mathbb{C}$ is parabolic.

3. The goal of this problem is to prove the following result that I stated in class. If D is a domain and P is a boundary point for which there is an analytic arc that ends at P and intersects $\overline{D}$ only at P, then there is a barrier at P, i.e., P is regular.

(a) First suppose that $D \subset \mathbb{C}$, $P = 0$, and the analytic arc is a line segment with one endpoint at the origin. Show there is a barrier in this case. Hint: $\sqrt{z}$.
(b) Reduce the general case to the above situation.

4. (taken from example on page 182 of Farkas and Kra.) Let D be a domain in $\mathbb{C}$ with $\overline{D}$ compact and ∂D regular. Let $z_0 \in D$. Let γ be the solution to the Dirichlet problem for D with boundary data $\gamma(z) = log|z - z_0|$ for $z \in \partial D$. So γ is harmonic on D and continuous on $\overline{D}$. Define $g(z) = \gamma(z) - log|z - z_0|$. Show that g is the Green's function on D with singularity at z_0.

5. (a) Find the Green's function for $\mathbb{D}$ with singularity at 0.
(b) Find the Green's function for $\mathbb{D}$ with singularity at a general $w \in \mathbb{D}$.

6. The goal of this problem is to prove the following result that I stated in class: It u is harmonic and bounded in $\{0 < |z| < 1\}$, then u can be extended to a harmonic function on $\{|z| < 1\}$.
(a) Hint : coming.