January 18, 2010

Syllabus

Caution: This is a work in progress and may change significantly. I have posted it to give an overall flavor of what the course is about. The section headings below are just copied from Farkas and Kra which I plan to follow pretty closely.

1. Riemann surfaces

We start with a heursitic look at Riemann's original motivation for Riemann sufaces. Then we define them, give examples and consider holomorphic maps between them. Next we review some differential geometry, algebraic topology and integration on manifolds (surfaces mainly).

1.1 Definitions and examples
1.2 Topology of Riemann surfaces
1.3 Differential forms

2. Existence theorems

It is not obvious a priori that there are any meromorphic functions on Riemann surfaces. The main goal of this chapter is to construct such functions. This is done by first showing there are meromorphic differentials. This chapter has a strong analysis flavor.

2.1 Hilbert space theory - review
2.2 Weyl's lemma
2.3 Hilbert space of square integrable forms
2.4 Harmonic differentials
2.5 Meromorphic functions and differentials

3. Compact Riemann surfaces

This chapter proves three important theorems on compact Riemann surfaces: the Riemann-Roch theorem, Abel's theorem, and the Jacobi inversion theorem. The statements of these theorems require developing some machinery, but we note a couple of the many important consequence of these theorems. The Riemann-Roch theorem allows us to compute the dimensions of certain vector spaces of meromorphic functions on a compact Riemann surface. One corrollary is that the only compact Riemann surfaces with genus zero (up to conformal equivalence) is the Riemann sphere. A corrollary of Abel's theorem is that every Riemann surfaces with genus one must be a torus. This chapter also studies hyperelliptic Riemann surfaces. These are two sheeted branched coverings of the sphere.

4. Uniformization

One of the main results of this chapter is the classification of Riemann surfaces. Up to conformal equivalence there are only three simply connected Riemann surfaces: $\mathbb{C}, \hat{\mathbb{C}}, \mathbb{H}$. For surfaces that are not simply connected, we look at the universal covering space. It is simply connected and so is one of the above three. If it is $\hat{\mathbb{C}}$ then it turns out that the original surface must be $\hat{\mathbb{C}}$. And if it is $\mathbb{C}$ then the original surface is either $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ or a torus. These are referred to as the exceptional Riemann surfaces. If a Riemann surface M is something else, the covering space is $\mathbb{H}$ and M is $\mathbb{H}/\Gamma$ where Γ is a subgroup of the group of Moibius transformations that map $\mathbb{H}$ onto $\mathbb{H}$. In fact, Γ is isomorphic to $\pi_1()$, the fundamental group of M.