Math 523a - Solutions to Final - Fall 2012

1. Let x_n be a sequence in $\mathbb{R}$. Define

$$f(x) = \sum_{n=1}^{\infty} \exp(-n^2|x - x_n|)$$

Prove that $f(x) < \infty$ a.e.

Solution: We will show that $\int f(x)\,dx < \infty$. This implies that $f(x)$ is finite a.e. Applying the monotone converge theorem to

$$g_n(x) = \sum_{k=1}^{n} \exp(-k^2|x - x_k|)$$

we conclude that

$$\begin{aligned}
\int f(x)\,dx &= \int \lim_{n\to\infty} g_n(x)\,dx = \lim_{n\to\infty} \int g_n(x)\,dx \\
&= \sum_{n=1}^{\infty} \int \exp(-n^2|x - x_n|)\,dx = \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty
\end{aligned}$$

2. In this problem X is $[0,\infty) \times [0,\infty)$ and m is two dimensional Lebesgue measure.

(a) Let $f(x, y) = \exp(-(x + 1)^2 y)$. Prove that f is integrable on X.

(b) Let $\lambda_n > 0$ with $\lambda_n \to 0$ as $n \to \infty$. Find

$$\lim_{n\to\infty} \int_X \exp(-(x + 1)^2(y + \lambda_n))\,dm$$

Your answer should be a number and you should prove it.

Solution: (a) By Tonelli's theorem,

$$\begin{aligned}
\int_X f(x, y)dm &= \int_0^{\infty} \left[\int_0^{\infty} \exp(-(x + 1)^2 y)dy\right] dx \\
&= \int_0^{\infty} \frac{1}{(x + 1)^2} dx = 1
\end{aligned}$$

(b) Let $f_n(x,y) = \exp(-(x+1)^2(y+\lambda_n))$ and $f(x,y) = \exp(-(x+1)^2 y)$ as before. Then $f_n(x,y) \le f(x,y) \in L^1$. As $n \to \infty$, $f_n(x,y)$ converges to $f(x,y)$ pointwise. So by the dominated convergence theorem, $\int_X f_n \, dm$ converges to $\int_X f \, dm$, which we computed in part (a) to be 1.

3. Give an example of an increasing function $F(x)$ on $[0,1]$ such that for all $0 < a < b < 1$ we have

$$\int_a^b F'(x)\, dx < F(b) - F(a)$$

You should justify your answer.

Solution: Let q_n be an enumeration of the rationals in $(0,1)$. Let λ be the measure

$$\lambda = \sum_{n=1}^{\infty} 2^{-n} \delta_{q_n}$$

This is the Borel measure associated with the increasing function

$$F(x) = \sum_{n: q_n \le x} 2^{-n}$$

Since λ is mutually singular with Lebesgue measure, $F'(x) = 0$ a.e. For any $0 < a < b < 1$ there is a rational q_n in (a,b) so $F(b) - F(a) \ge 2^{-n}$.

4. Let $g \ge 0$ be a Lebesgue measurable function on $\mathbb{R}$. Let m denote Lebesgue measure on $\mathbb{R}$ and define another measure by $\mu = gm$, i.e.,

$$\mu(E) = \int_E g\, dm$$

Find the Lebesgue-Radon-Nikodym decomposition of m with respect to μ, i.e., express m as $f\mu + \lambda$ where f is a non-negative measurable function and $\lambda \perp \mu$.

Solution: Let $N = \{x : g(x) = 0\}$. Let $\lambda(E) = m(E \cap N)$. Let $f(x) = 1/g(x)$ on N^c and $f(x) = 0$ on N. Let $\nu = f\mu$. Then ν is absolutely continous with respect to μ. Since $\lambda(N^c) = 0$ and $\mu(N) = 0$, we see that $\lambda \perp \mu$. To see that $m = \nu + \lambda$, note that $fg = 1$ on N^c, while $fg = 0$ on N, so for any Borel set E,

$$m(E) = m(E \cap N^c) + m(E \cap N) = \int_{E \cap N^c} fgdm + \lambda(E) = \int_E fgdm + \lambda(E)$$

We showed in a homework problem that $\int_E fgdm = \int_E fd\mu = \nu(E)$. So $m(E) = \nu(E) + \lambda(E)$.

5. We take our measurable space to be $\mathbb{N}$ with the σ-algebra of all subsets of $\mathbb{N}$. We define a measure by

$$\mu(E) = \sum_{k \in E} 2^{-k} \quad for \quad E \subset \mathbb{N}$$

Let f_n and f be functions on $\mathbb{N}$, all of which are integrable with respect to μ. Consider the three modes of convergence:

(a) f_n converges to f point-wise

(b) f_n converges to f in measure

(c) f_n converges to f in L^1 norm.

There are six possible implications ((a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c),etc). Prove the ones that are true and give counterexamples for those that are false.

Solution: Note that for an integrable g,

$$\int g \, d\mu = \sum_{n \in \mathbb{N}} g(n) 2^{-n}$$

Also note that μ is a finite measure. We proved in class that for a finite measure, point-wise convergence a.e. implies convergence in measure. So (a) $\Rightarrow$ (b).

Let f_n be the function that is 2^n at $k = n$ and is 0 everwhere else. Then $\int f_n d\mu = 1$. But $f_n \to 0$ both point wise and in measure. This shows (a) does not imply (c) and (b) does not imply (c).

We claim (c) $\Rightarrow$ (a). Fix a k. Then

$$|f_n(k) - f(k)| \leq 2^k \int |f_n - f| \, d\mu$$

We have shown (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b), so (c) $\Rightarrow$ (b). Note that we also know that (c) implies (b) since convergence in L^1 always implies convergence in measure.

Finally we show (b) implies (a). Fix a $k \in \mathbb{N}$. Let $\epsilon > 0$. Then the convergence in measure means that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\})$ goes to 0 as $n \to \infty$. So there is N such that $n \geq N$ implies it is less than 2^{-k}. But $\mu(k) = 2^{-k}$, so k cannot be in $\{x : |f_n(x) - f(x)| \geq \epsilon\}$ for $n \geq N$. So $|f_n(k) - f(k)| < \epsilon$ for $n \geq N$.