Math 523a - Homework 1

1. Let E_n be a sequence of subsets of some set X. Define
$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$$
The characteristic function, $\chi_E(x)$, of a set E is the function that is 1 when $x \in E$ and is 0 when $x \notin E$. Letting $E = \limsup_{n\to\infty} E_n$, prove that
$$\chi_E(x) = \limsup_{n\to\infty} \chi_{E_n}(x)$$

2. For each of the following prove that it is countable or that it is not:
   (a) The set of binary sequences that converge. (A binary sequence is a sequence containing only 0's and 1's.)
   (b) The set of sequence of rational numbers that converge.
   (c) The set of sequences of rational numbers that are eventually constant.

3. Let $a_{n,m} \geq 0$ for $n, m \in \mathbb{N}$. Using only what we did in class (or in the book), prove that
$$\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} a_{n,m}\right] = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} a_{n,m}\right]$$
Hint: show both sides are equal to the infinite sum we defined in class.

4. Let (X, ρ) be a metric space. Fix $x_0 \in X$. Define $f(x) = \rho(x, x_0)$.
   (a) Prove f is continuous on X.
   (b) Is f uniformly continuous on X ? Prove your answer.

5. Let l^∞ be the set of bounded sequences of real numbers. We define a metric on it by
$$\rho((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) = \sup_{1\leq n<\infty} |x_n - y_n|$$
Let U be the set of sequences in l^∞ that do not converge. Prove U is open.