Math 523a - Homework 2

1. (problem 4, p. 14 in Folland). Let $\mathcal{A}$ be an algebra. We say $\mathcal{A}$ is closed under increasing countable unions if for every sequence $E_n \in \mathcal{A}$ with $E_n \subset E_{n+1}$ for $n = 1, 2, \cdots$, we have $\cup_{n=1}^{\infty} E_n \in \mathcal{A}$. Prove that $\mathcal{A}$ is a σ-algebra if and only if it is closed under countable increasing unions.

2. (problem 5, p. 14 in Folland). For any collection $\mathcal{E}$ of subsets of X, let $\mathcal{M}(\mathcal{E})$ denote the σ-algebra generated by $\mathcal{E}$. Prove that

$$\mathcal{M}(\mathcal{E}) = \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$$

where the union is over all countable subsets $\mathcal{F}$ of $\mathcal{E}$.

3. Prove that the σ-algebra of Borel sets in $\mathbb{R}^2$ is generated by the collection of open balls in $\mathbb{R}^2$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E_n \in \mathcal{M}$ for $n = 1, 2, \cdots$. Prove that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty \quad \Rightarrow \quad \mu(\limsup E_n) = 0$$

Recall that $\limsup E_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$.

5. (problem 11, p. 27 in Folland) Let $(X, \mathcal{M}, \mu)$ be a **finite** measure space.

   (a) Prove that if $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$ then $\mu(E) = \mu(F)$.

   (b) Define $E \sim F$ if $\mu(E \Delta F) = 0$. Prove that $\sim$ is an equivalence relation.

   (c) Prove that $\rho(E, F) = \mu(E \Delta F) = 0$ defines a metric on the set of equivalence classes in $\mathcal{M}$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that it is *complete* if for every $N \in \mathcal{M}$ such that $\mu(N) = 0$, every subset of N is in $\mathcal{M}$ (and so has measure 0). If the measure space is not complete, there is a natural way to "complete it." Define

$$\begin{aligned} \mathcal{N} &= \{N \in \mathcal{M} : \nu(N) = 0\} \\ \overline{\mathcal{M}} &= \{E \cup F : E \in \mathcal{M}\, and\, F \subset N\, for\, some\, N \in \mathcal{N}\} \end{aligned}$$

Prove that $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension of μ to a complete measure on $\overline{\mathcal{M}}$.