Math 523a - Homework 3

1. Problem 18, p. 32 in Folland.

2. Problem 19, p. 33 in Folland.

3. Let $\mathcal{E}$ be the collection of open intervals (a, b). Define ρ on E by $\rho((a, b)) = (b - a)^2$. Let μ^* be the outer measure induced by ρ, i.e.,

$$\mu^*(E) = \inf\{\sum_{n=1}^{\infty} \rho((a_n, b_n)) : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

   Prove that for every set E, $\mu^*(E) = 0$.

4. Let $\mathcal{E}$ be the collection of open intervals (a, b). Define ρ on E by $\rho((a, b)) = \sqrt{b - a}$. Let μ^* be the outer measure induced by ρ.

   (a) Compute $\mu^*((a, b))$. (Hint: $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$.)

   (b) Prove that no open interval is μ^*-measurable.

   (c) (optional) Prove that if E is a bounded measurable set, then $\mu^*(E) = 0$.

5. We defined the Cantor function in class. It is a continuous function F on $[0, 1]$ with $F(0) = 0$, $F(1) = 1$ that is constant on the complement of the Cantor set. We can extend it to all of $\mathbb{R}$ by defining it to be 0 for $x < 0$ and 1 for $x > 1$. Let μ_C be the associated Borel measure. Let m be Lebesgue measure.

   (a) Prove that there is a Borel set S such that $m(S) = 0$ and $\mu_C(E) = \mu_C(S \cap E)$ for all Borel sets E.

   (b) Prove that there is a Borel set U such that $\mu_C(U) = 0$ and $m(E) = m(S \cap E)$ for all Borel sets E.

6. Problem 26, p. 39 in Folland.

7. Problem 30, p. 40 in Folland.

8. Problem 31, p. 40 in Folland.

You should do problem 28, p. 39 in Folland, but don't turn it in.