**Math 523a - Homework 3 - selected solutions**

**2. Problem 19, p. 32 in Folland**

First suppose $E$ is measurable. Then $\mu^*(E) = \mu(E)$. Since $\mu(X) < \infty$,

$$\mu_*(E) = \mu(X) - \mu^*(E^c) = \mu(X) - \mu(E^c) = \mu(X \setminus E^c) = \mu(E)$$

So $\mu_*(E) = \mu^*(E)$.

Now suppose $\mu_*(E) = \mu^*(E)$. By part (a) of previous problem we can find $A_n \in \mathcal{A}_\sigma$ with $E \subset A_n$ and $\mu(A_n) \leq \mu^*(E) + 1/n$. Let $A = \cap_n A_n$. Note that $A$ is measurable. Then $E \subset A$ and $\mu^*(E) = \mu(A)$. Similarly, there is a measurable set $B$ with $E^c \subset B$ and $\mu^*(E^c) = \mu(B)$. Now

$$\mu(B^c) = \mu(X) - \mu(B) = \mu(X) - \mu(E^c) = \mu_*(E) = \mu^*(E) = \mu(A)$$

We have $B^c \subset E \subset A$, so $\mu(A \setminus B^c) = \mu(A) - \mu(B^c) = 0$. Since the measure is complete, every subset of $A \setminus B^c$ is measurable. In particular, $E \setminus B^c$ is measurable. Since $E = B^c \cup (E \setminus B^c)$, this proves $E$ is measurable.

**8. Problem 31, p. 40 in Folland:** By the previous problem there is a interval $I$ such that $m(I \cap E) \geq \frac{3}{4} m(I)$. Let $F = E \cap I$. Since $(F - F) \subset (E - E)$, it suffice to show $F - F$ contains an interval about 0. Note that $F \subset I$ and $m(F) \geq \frac{3}{4} m(I)$. Thus $m(I \setminus F) = m(I) - m(F) \leq \frac{1}{4} m(I)$.

First we claim that if $F \cap (F+x) \neq \emptyset$, then $x \in F - F$. Let $y \in F \cap (F+x)$. Then $y \in F$ and there is a $z \in F$ such that $y = z + x$. So $x = y - z$, proving that $x \in F - F$.

Next we claim that if $|x| < \frac{1}{2} m(I)$, then $F \cap (F + x) \neq \emptyset$. (This will complete the proof.) First note that the condition on $x$ implies that

$$m(I \cap (I + x)) = m(I) - |x| > \frac{1}{2} m(I).$$

Now some set theory:

$$\begin{aligned} I \cap (I + x) &= [F \cap (I + x)] \cup [(I \setminus F) \cap (I + x)] \\ &= [F \cap (F + x)] \cup [F \cap ((I \setminus F) + x)] \cup [(I \setminus F) \cap (I + x)] \end{aligned}$$

We have

$$m(F \cap ((I \setminus F) + x)) \leq m((I \setminus F) + x)) = m(I \setminus F) \leq \frac{1}{4} m(I)$$

and

$$m((I \setminus F) \cap (I + x)) \leq m(I \setminus F) \leq \frac{1}{4} m(I)$$

So

$$\frac{1}{2} m(I) < m(I \cap (I + x)) \leq m(F \cap (F + x)) + \frac{1}{2} m(I)$$

Thus $m(F \cap (F + x)) > 0$ which of course implies $F \cap (F + x) \neq \emptyset$.