Math 523a - Homework 4

1. Prove proposition 2.11 on p. 47 of the book. **NB:** The definition I gave in class of a.e. was not quite right. We say that some property which depends on x holds a.e., if there is a measurable set E with measure zero such that the property holds for all $x \notin E$. This is not the same as saying the set of x for which the property does not hold has measure zero, since the set where the property fails may not be a measurable set.

2. Let $(X, \mathcal{M})$ be a measurable space. Let $f : X \to \mathbb{R}$ be a non-negative, measurable function. Define a subset of $X \times \mathbb{R}$ by

$$E = \{(x,t) : f(x) \leq t\}$$

We use the product σ-algebra on $X \times \mathbb{R}$, i.e., $\mathcal{M} \otimes \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-field in $\mathbb{R}$. Prove that E is in $\mathcal{M} \otimes \mathcal{B}$.

3. (Folland, problem 14, p 52). Let $f \geq 0$ be integrable on $(X, \mathcal{M})$. Define λ on $\mathcal{M}$ by

$$\lambda(E) = \int_E f \, d\mu, \quad E \in \mathcal{M}$$

Prove λ is a measure on $(X, \mathcal{M})$ and for any integrable non-negative function g,

$$\int g \, d\lambda = \int f \, g \, d\mu$$

Hint: First prove it for simple f.

4. (Folland, problem 20, p 59) If f_n, g_n, f, g are non-negative integrable functions such that $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$ a.e., and $\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu$. Prove that $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

5. (Folland, problem 21, p 59) Suppose f_n and f are integrable and $f_n \to f$ a.e. Prove $\int |f_n - f| \, d\mu \to 0$ if and only if $\int |f_n| \, d\mu \to \int |f| \, d\mu$.

6. Let m be Lebesgue measure on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Define

$$F(x) = \int_{-\infty}^{x} f(t) \, dm(t)$$

(a) Prove that $F(x)$ is continuous.

(b) Prove that $F(x)$ is uniformly continuous.

7. (Folland, problem 25, p 59) Let $f(x) = x^{-1/2}\chi_{[0,1]}(x)$ on $\mathbb{R}$. Let $\{r_n\}_{n=1}^{\infty}$ be the rationals in $\mathbb{R}$. Define

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$$

   (a) Prove g is integrable.

   (b) Prove g is discontinuous at every point and unbounded on every interval, and this is true for any function h such that $g = h$ almost everywhere.

   (c) Prove that $g^2 < \infty$ a.e., but g^2 is not integrable.

8. (Folland, problem 28 a, p 59) Compute the following limit and rigorously justify your computation.

$$\lim_{n\to\infty} \int_0^{\infty} \left[1 + \frac{x}{n}\right]^{-n} \sin(\frac{x}{n})\, dx$$

9. For what value of α is the following limit finite and non-zero? Compute, with justification, the value of the limit for this choice of α.

$$\lim_{n\to\infty} n^{\alpha} \int_0^{\infty} \frac{\sin(x)}{(1+x^2)^n} dx.$$