Math 523a - Homework 5

1. Folland, problem 46, p 68.

2. Folland, problem 59, p 77.

3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $f : X \to \mathbb{R}$ be a non-negative, measurable function. Define a subset of $X \times \mathbb{R}$ by

$$E = \{(x,t) : f(x) \geq t \geq 0\}$$

We use the product σ-algebra on $X \times \mathbb{R}$, i.e., $\mathcal{M} \otimes \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-field in $\mathbb{R}$. In the last homework you proved that E is in $\mathcal{M} \otimes \mathcal{B}$. Let m denote Lebesgue measure on $\mathbb{R}$. Then $\mu \times m$ is a measure on $(X \times \mathbb{R}, \mathcal{M} \otimes \mathcal{B})$. Prove that

$$\mu \times m(E) = \int f \, d\mu = \int_{[0,\infty)} F(t) dm$$

where $F(t) = \mu(\{x : f(x) \geq t\})$. Hint: $\mu \times m(E) = \int \chi_E \, d\mu \times m$.

4. Let μ be a **finite** Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} \sqrt{|x|} \, d\mu < \infty$. Prove that there is a set $N \subset \mathbb{R}$ of Lebesgue measure zero such that for $s \notin N$,

$$\int_{\mathbb{R}} |s-t|^{-1/2} \, d\mu(t) < \infty$$

Hint: Let $L > 0$. Prove that the above is finite when $|s| \leq L$, except for a set of Lebesgue measure zero. For this consider

$$\int_{[-L,L]} \left[\int_{\mathbb{R}} |s-t|^{-1/2} \, d\mu(t) \right] dm(s)$$

5. Let m denote Lebesgue measure on $\mathbb{R}^n$. For this problem theorem 2.49 on p. 78 may be useful. (We didn't prove this in class, but you can use it.)

   (a) Let

$$f(x_1, x_2, \cdots, x_n) = \left[\sum_{i=1}^{n} x_i \right]^{-p}$$

   Determine (with justification) for what values of p we have $f \in L^1([0,1]^n, m)$.

(b) Let

$$g(x_1, x_2, \cdots, x_n) = \left[1 + \sum_{i=1}^{n} x_i\right]^{-p}$$

Determine (with justification) for what values of p we have $f \in L^1([0, \infty)^n, m)$.