Math 523a - Homework 7

1. This is related to the homework problem in the last set about conditional expectation. Let $(X, \mathcal{M}, P)$ be measure space with $P(X) = 1$. Such a measure is called a probability measure and it is traditional to use capital roman letters like P to denote probability measures. Let $\{E_n\}_{n=1}^N$ be a collection of disjoint measurable sets whose union is X. The collection can be finite (in which case N is finite) or countable infinite (in which case N is ∞.) In probability, measurable sets are called events, and measurable functions are called random variables. If f is integrable, then $\int f dP$ is called the expectation or expected value of f. In probability it is denoted $E[f]$. Let $\mathcal{N}$ be the σ-algebra generated by $\{E_n\}_{n=1}^N$.

   (a) Give an explicit characterization of $\mathcal{N}$. In particular, how many sets are in $\mathcal{N}$ if N is finite?

   (b) Given an explicit characterization of the functions that are measurable with respect to $\mathcal{N}$.

   (c) If $A, B \in \mathcal{M}$ with $P(B) > 0$, the conditional probability of A given B is defined to be $P(A|B) = P(A \cap B)/P(B)$. As we have seen before, $A \to P(A|B)$ defines a probability measure. If f is integrable, the expectation of f with respect to this probability measure is called the conditional expectation of f with respect to B. We denote if by $E[f|B]$. Show that

   $$E[f|B] = \frac{E[1_B f]}{P(B)}$$

   (d) Let $f \in L^1(P)$. In the last homework set you showed that for any sub σ-algebra $\mathcal{N}$ there is another function $g \in L^1(P)$ which is $\mathcal{N}$ measurable and approximates f in the sense that

   $$\int_E f\, dP = \int_E g\, dP, \quad if\ E \in \mathcal{N}$$

   g is called the conditional expectation of f given $\mathcal{N}$ and denoted $E[f|\mathcal{N}]$ in probability. Note that $E[f|\mathcal{N}]$ is a function while $E[f|B]$ is a number. Give an explicit formula for $E[f|\mathcal{N}]$ when $\mathcal{N}$ is the σ-algebra defined above.

2. Folland, problem 25, p 100.
3. Folland, problem 26, p 100.
4. Folland, problem 28, p 107.
5. Folland, problem 33, p 108.
6. Folland, problem 36, p 108.
7. Folland, problem 37, p 108.