Math 523a - Midterm - Take home part

Due Monday, Oct 22 at 10 am.
Rules: You may not talk to anyone about the exam. If the problem is unclear you can ask me for clarification, but I will not give hints. You may refer to your class notes and Folland. But you cannot use any other books or the web.
Exposition is important. I will take off points if your solution is not clearly explained or contains irrelevant arguments.

1. Recall that for two sets E, F, we define $E \Delta F = (E \setminus F) \cup (F \setminus E)$. And for a subset E of $\mathbb{R}$ we define $E + x = \{y + x : y \in E\}$. Let m be Lebesgue measure on the real line. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $m(E) < \infty$. Find the limit

$$\lim_{x \to \infty} m(E \Delta (E + x))$$

   You should prove your answer.

2. Let $(X, \mathcal{M})$ be a measurable space, and (Y, d) a **separable** metric space. Equip Y with the Borel σ-algebra. Let $f_n : X \to Y$ be measurable. Let $E \subset X$ be the set of x such that $f_n(x)$ is a Cauchy sequence in (Y, d). Prove that E is measurable. (You may not assume that Y is a complete metric space.)

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be a non-negative function in $L^1(X, \mathcal{M}, \mu)$ such that $\mu(\{x : f(x) \leq 1\}) < \infty$.

   (a) Show that for positive integers n, $f^{1/n}$ is in L^1.

   (b) Find

$$\lim_{n \to \infty} \int f^{1/n} \, d\mu$$

   You should prove your answer.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. For real-valued measurable functions f, g on X, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}\, d\mu$$

Prove that $f_n \to f$ in measure if and only if $\rho(f_n, f) \to 0$.

5. Let $C[0,1]$ be the set of real-valued continuous functions on $[0,1]$. Define a metric on this set by

$$\rho(f,g) = \int_{[0,1]} |f-g|\, dm$$

where m is Lebesgue measure on $[0,1]$. Let

$$A = \{f \in C[0,1] : |f(x)| \leq 1\, \forall x\}$$

Find the interior of A. You should prove your answer.