Math 523a - Midterm - Take home part solutions

1. Recall that for two sets E, F, we define $E\Delta F = (E \setminus F) \cup (F \setminus E)$. And for a subset E of $\mathbb{R}$ we define $E + x = \{y + x : y \in E\}$. Let m be Lebesgue measure on the real line. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $m(E) < \infty$. Find the limit

$$\lim_{x\to\infty} m(E\Delta(E + x))$$

You should prove your answer.

Solution We will prove that the limit is $2m(E)$. We start by observing that

$$\begin{aligned} m(E\Delta(E + x)) &\leq m(E \setminus (E + x)) + m((E + x) \setminus E) \\ &\leq m(E) + m(E + x) = 2m(E) \end{aligned}$$

since m is translation invariant. Thus

$$\limsup_{x\to\infty} m(E\Delta(E + x)) \leq 2m(E)$$

Now let $\epsilon > 0$. By the regularity of m and the fact that $m(E) < \infty$, there is a compact set $F \subset E$ with $m(F) \geq m(E) - \epsilon$. So $\mu(E \setminus F) \leq \epsilon$. Now

$$(E \setminus (E + x)) \cup ((E + x) \setminus E) \supset (F \setminus (E + x)) \cup ((F + x) \setminus E)$$

Since F is compact it is bounded. So F and $F + x$ are disjoint for large enough x. So $F \setminus (E + x)$ and $(F + x) \setminus E$ are disjoint for large enough x. So

$$\begin{aligned} & m(E\Delta(E + x)) \geq m(F \setminus (E + x)) + m((F + x) \setminus E) \\ = \; & m(F) - m(F \cap (E + x)) + m(F + x) - m((F + x) \cap E) \quad (1) \end{aligned}$$

Note that $m(F)+m(F+x) = 2m(F) \geq 2m(E)-2\epsilon$. Since F is bounded we can find M so that $F \subset [M, -M]$. Then $F + x \subset [M + x, -M + x]$. So

$$m((F + x) \cap E) \leq m([M + x, -M + x] \cap E) \leq m([M + x, \infty) \cap E)$$

If x_n is any sequence increasing to ∞, then the sets $[M + x_n, \infty) \cap E$ are a decreasing sequence of sets whose intersection is empty. Since $m(E) < \infty$ we can conclude from continuity of the measure m that $m([M + x_n, \infty) \cap E) \to 0$. And so $\lim_{x\to\infty} m((F + x) \cap E) = 0$. By translation invariance, $m(F \cap (E+x)) = m((F-x) \cap E)$. An argument similar to the preceeding shows this also goes to zero as $x \to \infty$. Thus taking the lim inf of eq. (1) we have

$$\liminf_{n\to\infty} m(E\Delta(E + x)) \geq 2m(E) - 2\epsilon$$

This is true for all $\epsilon > 0$, so the lim inf must be $\geq 2m(E)$ Combining all this

$$\begin{aligned}\limsup_{n\to\infty} m(E\Delta(E + x)) &\leq 2m(E) \leq \liminf_{n\to\infty} m(E\Delta(E + x)) \\ &\leq \limsup_{n\to\infty} m(E\Delta(E + x))\end{aligned}$$

Thus the lim inf and lim sup are equal and the limit is $2m(E)$.

2. Let $(X, \mathcal{M})$ be a measurable space, and (Y, d) a metric space. Equip Y with the Borel σ-algebra. Let $f_n : X \to Y$ be measurable. Let $E \subset X$ be the set of x such that $f_n(x)$ is a Cauchy sequence in (Y, d). Prove that E is measurable. (You may not assume that Y is a complete metric space.)

   Solution $f_n(x)$ is Cauchy if $\forall\epsilon > 0$ there exist a positive integer N such that $n, m \geq N \Rightarrow d(f_n(x), f_m(x)) < \epsilon$. If this hold for $\epsilon = 1/k$ for all positive integers k, then it holds for all $\epsilon > 0$. So

$$E = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{x : d(f_n(x), f_m(x)) < \frac{1}{k}\}$$

   Since countable unions and countable intersections of measurable sets are measurable, it suffices to show $\{x : d(f_n(x), f_m(x)) < \epsilon\}$ is measurable for all n, m. Fix n, m and look at the function $\phi(x) = d(f_n(x), f_m(x))$ from X to the reals. The set in question is $\phi^{-1}([0, \epsilon))$. So if we can show ϕ is measurable we are done. We can write $\phi = G \circ F$ where $F : X \to Y \times Y$ by $F(x) = (f_n(x), f_m(x))$ and $G : Y \times Y \to \mathbb{R}$ by $G(y, z) = d(y, z)$. Let $\mathcal{B}_Y$ be the Borel sets in Y. By a theorem from class F is measurable from $(X, \mathcal{M})$ to $(Y \times Y, \mathcal{B}_Y \otimes \mathcal{B}_Y)$ if (and only if) its component functions $f_n(x)$ and $f_m(x)$ are measurable. So F is measurable in this sense. G is continuous if we use the product metric on $Y \times Y$, and so is measurable if we use the Borel sets $\mathcal{B}_{Y\times Y}$ in $Y \times Y$ that come from the product metric. Since Y is separable, $\mathcal{B}_{Y\times Y} = \mathcal{B}_Y \otimes \mathcal{B}_Y$. So ϕ is the composition of two measurable functions and so is measurable.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be a non-negative function in $L^1(X, \mathcal{M}, \mu)$ such that $\mu(\{x : f(x) \leq 1\}) < \infty$.

   (a) Show that for positive integers n, $f^{1/n}$ is in L^1.

   (b) Find

$$\lim_{n\to\infty} \int f^{1/n}\, d\mu$$

   You should prove your answer.

   Solution (a) If $a > 1$ then $a^{1/n} \leq a$. So when $f(x) > 1$ we can bound $f(x)^{1/n}$ by $f(x)$. When $f(x) \leq 1$ we just bound it by 1. So if we let $g(x) = 1 + f(x)$, then $f^{1/n}(x) \leq g(x)$. Since $\mu(X) < \infty$, 1 is in L^1. So g is in L^1.

   (b) We want to use the dominated convergence theorem. When $f(x) > 0$, $f(x)^{1/n} \to 1$. And when $f(x) = 0$, $f(x)^{1/n} = 0$. So the sequence $f^{1/n}$ converges pointwise to χ_E where $E = \{x : f(x) > 0\}$. The function g in the previous part provides a dominating function. So we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int f^{1/n}\, d\mu = \int \chi_E d\mu = \mu(E)$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. For real-valued measurable functions f, g on X, define

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|}\, d\mu$$

   Prove that $f_n \to f$ in measure if and only if $\rho(f_n, f) \to 0$.

Solution Note that for $a \geq 0$, $a/(1+a) < 1$. So the integrand in $\rho(f,g)$ is pointwise bounded by 1. For $\epsilon > 0$, define

$$E_{\epsilon,n} = \{x : |f_n(x) - f(x)| \geq \epsilon\}$$

So f_n converges to f in measure if and only if for all $\epsilon > 0$ we have $\lim_{n\to\infty} \mu(E_{\epsilon,n}) = 0$.

Now suppose f_n converges to f in measure. Then

$$\begin{aligned}
\rho(f_n, f) &= \int_{E^c_{\epsilon,n}} \frac{|f-g|}{1+|f-g|}\, d\mu + \int_{E_{\epsilon,n}} \frac{|f-g|}{1+|f-g|}\, d\mu \\
&\leq \int_{E^c_{\epsilon,n}} \epsilon d\mu + \int_{E_{\epsilon,n}} 1\, d\mu \\
&\leq \epsilon\mu(X) + \mu(E_{\epsilon,n})
\end{aligned}$$

Taking lim sup of both sides, we conclude $\limsup_n \rho(f_n, f) \leq \epsilon\mu(X)$. This holds for all $\epsilon > 0$, so $\limsup_n \rho(f_n, f) = 0$. Since $\rho(f_n, f) \geq 0$, this implies $\lim_n \rho(f_n, f) = 0$.

Now suppose $\lim_n \rho(f_n, f) = 0$. Note that $x \to x/(1+x)$ is an increasing function on $[0, \infty)$. So $y \geq \epsilon$ if and only if $y/(1+y) \geq \epsilon/(1+\epsilon)$. So on $E_{\epsilon,n}$, $|f-g|/(1+|f-g|) \geq \epsilon/(1+\epsilon)$. So

$$\rho(f_n, f) \geq \int_{E_{\epsilon,n}} \frac{|f-g|}{1+|f-g|}\, d\mu \geq \int_{E_{\epsilon,n}} \frac{\epsilon}{1+\epsilon}\, d\mu = \frac{\epsilon}{1+\epsilon}\mu(E_{\epsilon,n})$$

So $\lim_n \mu(E_{\epsilon,n}) = 0$, which says that f_n converges to f in measure.

5. Let $C[0,1]$ be the set of real-valued continuous functions on $[0,1]$. Define a metric on this set by

$$\rho(f,g) = \int_{[0,1]} |f-g|\, dm$$

where m is Lebesgue measure on $[0,1]$. Let

$$A = \{f \in C[0,1] : |f(x)| \leq 1\, \forall x\}$$

Find the interior of A. You should prove your answer.

Solution Given $\epsilon > 0$ let g_ϵ be a continuous function with $g(1/2) = 3$ and $\int |g| \, dm < \epsilon$. If $f \in A$, then a ball of radius ϵ centered at f will contain $f + g$. But $|(f+g)(1/2)| \geq |g(1/2)| - |f(1/2)| \geq 3 - 1 = 2$. So $f + g \notin A$. So there is no open ball centered at f which is contained in A. So the interior of A is empty.