Final Exam

Math 523a – Real Analysis – Prof. Venkataramani

This was a three hour exam and the students had to do any six of the eight questions.

(1) $f_n : (0,1) \to \mathbb{R}$ is a sequence of continuous functions. Show that the set of points x in $(0,1)$ such that $f_n(x)$ converges (pointwise convergence) is a Borel set.

(2) Let $\mathcal{E}$ be the set of all open intervals on $\mathbb{R}$. Let $\rho((a,b)) = (b-a)^2$ and μ^* be the outer measure induced by ρ. Show that $\mu^*(\mathbb{R}) = 0$, and hence $\mu^*(A) = 0$ for all $A \subseteq \mathbb{R}$.

(3) For what values(s) of α is the following limit finite and non-zero? Compute the value of the limit for this choice of α.

$$\lim_{n\to\infty} n^\alpha \int_0^n \log\left(1 - \frac{x}{n}\right) e^{-nx}\, dx$$

(4) f is a nonnegative, Lebesgue measurable function. Assume that $x^2\left[e^{f(x)} - 1\right]$ is in $L^1(\mathbb{R})$. Show that for every $y > 0$, the set $M_y = \{x : f(x) > y\}$ is measurable, and the function x^2 is integrable on M_y. Express $\int x^2\left[e^{f(x)} - 1\right]$ in terms of $\alpha(y) = \int_{M_y} x^2 dx$.

(5) $f_n \to f$ in measure, and there is a function $g \in L^1$ such that $|f_n| \leq g$ for all n and for a.e x. Show that $f_n \to f$ in L^1.

(6) $X = A_1 \cup A_2 \cup \ldots \cup A_n$ is a finite partition of X into mutually disjoint sets. $\mathcal{M} = \mathcal{M}(A_1, A_2, \ldots, A_n)$ is the σ-algebra generated by the sets in the partition.

(i) Show that $\mathrm{card}(\mathcal{M}) = 2^n$.

(ii) μ is a finite measure on $\mathcal{M}$ with $\mu(A_i) > 0$ for all i. ν is a signed measure on $\mathcal{M}$ whose total variation $|\nu|(X)$ is finite. Show that $\nu \ll \mu$ and compute $\dfrac{d\nu}{d\mu}$. (Hint: The Radon-Nikodym derivative will be expressible in terms of $\mu(A_i)$ and $\nu(A_j)$ for $1 \leq i, j \leq n$.)

(7) $f_n : [0,1] \to \mathbb{R}$ is a sequence of non-negative continuous functions. $g = \sum_n f_n$. Show that

$$\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \{x : f_n(x) \leq e^{-n/k}\} = \left\{x : \limsup [f_m(x)]^{1/m} < 1\right\} \subseteq \{x : g(x) \neq +\infty\}.$$

(8) Let $f(x) = x^{-1}$ if $0 < x < 1, f(x) = 0$ otherwise. Let $\{r_n\}_1^{\infty}$ be an enumeration of the rationals and $g(x) = \sum_1^{\infty} 4^{-n} f(x - r_n)$. Show that g is not integrable on any interval, but the measure $\mu(E) = \int_E g$ is σ-finite.