Math 523b - Final - Spring 2013

Do four of the following five problems. Do not turn in five problems. If you turn in five problems, I will use your four lowest scores to compute your total.

WRITE YOUR NAME ON EACH SHEET YOU TURN IN AND DO NOT WRITE ON THE BACK

We use dx to denote Lebesgue measure.

1. Let X and Y be locally compact topological spaces. Prove that $X \times Y$ with the product topology is locally compact.

2. Let $1 < p < \infty$. Suppose $f_n \in L^p([0,1], dx)$ with $||f_n||_p \leq 1$ for all n. Suppose also that for all $\alpha \geq 0$

$$\lim_{n \to \infty} \int_0^1 f_n(x) e^{-\alpha x} \, dx = 0$$

   Prove that f_n converges to 0 weakly in L^p. (You may assume all functions are real valued.)

3. Let M be a closed subspace of a Hilbert space H. Let F be a bounded linear functional on M. The Hahn Banach theorem says that there is an extension of F to a linear functional on all of H which has the same norm. Prove that such an extension is unique.

4. Let $C_0 = C_0(\mathbb{R})$ be the real-valued continuous functions on the real line that go to zero at $\pm\infty$. We use the usual sup norm. If $g \in L^1(\mathbb{R}, dx)$ is a real-valued Lebesgue integrable function, then we can define a bounded linear function ϕ_g on C_0 by

$$\phi_g(f) = \int f(x) g(x) \, dx$$

   Let F be the set of all such linear functionals:

$$F = \{\phi_g : g \in L^1(\mathbb{R}, dx)\}$$

   Prove that F is a closed subset of C_0^*.

5. Let X_n be an independent, identically distributed sequence of random variables with $X_1 \geq 0$ a.e. and $E[X_1] = \infty$. This implies that for all $c > 0$,

$$\sum_{n=1}^{\infty} P(X_1 \geq cn) = \infty$$

You may assume this fact without proving it.

(a) Prove that $P(X_n \geq c\,n \quad infinitely \quad often) = 1$ for all $c > 0$.

(b) Prove that for all $c > 0$

$$P(\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j \geq c) = 1$$

(c) Prove that

$$P(\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j = \infty) = 1$$