Math 523b - Final Exam Solutions- Spring 2013

Do four of the following five problems.

We use dx to denote Lebesgue measure.

1. Let X and Y be locally compact topological spaces. Prove that $X \times Y$ with the product topology is locally compact.

   Solution: Let $(x, y) \in X \times Y$. Since X and Y are locally compact there is a compact neighborhood U of x in X and a compact neighborhood V of y in Y. So $x \in int(U)$, $y \in int(V)$. By the definition of product toplogy, $int(U) \times int(V)$ is open in $X \times Y$. So $U \times V$ is a neighborhood of (x, y) in $X \times Y$.

   Since U is compact in X, U with the relative topology is a compact topological space. Likewise for V. So by Tychynoff's theorem, $U \times V$ with its product topology is compact. Some definition chasing shows that this product topology on $U \times V$ is the same as the relative topology it gets from the product topology on $X \times Y$. So $U \times V$ is compact in $X \times Y$ and so is a compact neighborhood of (x, y).

2. Let $1 < p < \infty$. Suppose $f_n \in L^p([0, 1], dx)$ with $||f_n||_p \leq 1$ for all n. Suppose also that for all $\alpha \geq 0$

$$\lim_{n \to \infty} \int_0^1 f_n(x) e^{-\alpha x} \, dx = 0$$

   Prove that f_n converges to 0 weakly in L^p. (You may assume all functions are real valued.)

   Solution: Let q be the conjugate exponent to p. To show f_n converges weakly to 0, we must show $\int_0^1 f_n g dx$ converges to 0 for all $g \in L^q$. We are given that this is true for g of the form $e^{-\alpha x}$. Let A be the set of functions of the form $\sum_{j=1}^n c_j \exp(-\alpha_j x)$ where n is any positive integer, $c_j \in \mathbb{R}$ and $\alpha_j \geq 0$. By linearity we have the desired convergence for $g \in A$.

   Since $\exp(-\alpha_1 x) \exp(-\alpha_2 x) = \exp(-(\alpha_1 + \alpha_2)x)$, A is an algebra. It contains the constant functions (take $\alpha = 0$). It contains e^{-x} which separates all point in $[0, 1]$. So by the Stone-Weierstrass theorem, A is dense in $C[0,1]$ with the sup norm. This implies it is dense in $C[0,1]$ with respect to the L^q norm since the measure is finite. And since $C[0,1]$ is dense in L^q, A is dense in L^q.

Now let $h \in L^q$ and let $\epsilon > 0$. Pick $g \in A$ such that $||h - g||_q < \epsilon$. Then by Holder's inequality,

$$|\int_0^1 f_n g dx - \int_0^1 f_n h dx| \leq ||f_n||_p ||g - h||_q \leq ||g - h||_q < \epsilon$$

So

$$\limsup_{n\to\infty} |\int_0^1 f_n(x) h(x)\, dx| \leq \epsilon$$

This holds for all $\epsilon > 0$, so the limsup is 0 and so

$$\lim_{n\to\infty} \int_0^1 f_n(x) h(x)\, dx = 0$$

3. Let M be a closed subspace of a Hilbert space H. Let F be a bounded linear functional on M. The Hahn Banach theorem says that there is an extension of F to a linear functional on all of H which has the same norm. Prove that such an extension is unique.

   Solution: We recall for $y \in H$, $x \to < x, y >$ defines a bounded linear functional on H whose norm is $||y||$. This gives an isometry between the dual of a Hilbert space and the Hilbert space itself. The closed subspace M is itself a Hilbert space. So we can restrict F to M and get a bounded linear functional on M. Then there is a unique $y \in M$ such that $F(x) =< x, y >$ for $x \in M$.

   Any extension of $\bar{F}$ of F to H must be of the form $\bar{F}(x) =< x, z >$ for some z in H. By the projection theorem z has a unique decomposition $z = z_1 + z_2$ where $z_1 \in M$ and $z_2 \in M^\perp$. Since it is an extension we must have $< x, y >=< x, z_1 + z_2 >=< x, z_1 >$ for all $x \in M$. So $< x, y - z_1 >= 0$ for all $x \in M$. But $y - z_1 \in M$, so $y - z_1$ must be 0, i.e., $z_1 = y$. Finally we note that the norm of the original F is $||y||$ while the norm of the extension is $||z_1 + z_2|| = ||y + z_2|| = \sqrt{||y||^2 + ||z_2||^2}$ So $||z_2|| = 0$, i.e., $z_2 = 0$. Thus the only possible extension is given by $z = y$.

4. Let $C_0 = C_0(\mathbb{R})$ be the real-valued continuous functions on the real line that go to zero at $\pm\infty$. We use the usual sup norm. If $g \in L^1(\mathbb{R}, dx)$ is a real-valued Lebesgue integrable function, then we can define a bounded linear function ϕ_g on C_0 by

$$\phi_g(f) = \int f(x)g(x)\,dx$$

Let F be the set of all such linear functionals:

$$F = \{\phi_g : g \in L^1(\mathbb{R}, dx)\}$$

Prove that F is a closed subset of C_0^*.

Solution One: The dual C_0^* is identified with the space of finite Borel measures on $\mathbb{R}$ in the usual way. For the linear functionals ϕ_g, the measure that represent the functional is $g(x)dx$. The norm of ϕ_g as a linear functional is the norm of the measure which is the L^1 norm of g. So if ϕ_{g_n} converges to some bounded linear functional ϕ in C_0^*, then g_n is a Cauchy sequence in L^1 and so by the completeness of L^1 converges to some $g \in L^1$. This means that ϕ_{g_n} converges to ϕ_g in C_0, and so F is closed.

Solution Two: Again, we identify C_0^* with the space of finite Borel measures. By the Randon-Nikodym theorem, F corresponds with the Borel measures that are absolutely continuous with respect to Lebesgue measure. The norm on C_0 corresponds to the total variation norm of the measures. So we need to show that if $||\mu_n - \mu|| \to 0$ for Borel measures μ_n and μ and $\mu_n << m$ for all n where m is Lebesgue measure, then $\mu << m$. Suppose E is a Borel set with $m(E) = 0$. Then $\mu_n(E) = 0$ for all n. Since

$$|\mu_n(E) - \mu(E)| = |(\mu_n - \mu)(E)| \leq ||\mu_n - \mu||$$

we conclude that $\mu(E) = 0$. So $\mu << m$ and thus F is closed.

5. Let X_n be an independent, identically distributed sequence of random variables with $X_1 \geq 0$ a.e. and $E[X_1] = \infty$. This implies that for all $c > 0$,

$$\sum_{n=1}^{\infty} P(X_1 \geq cn) = \infty$$

You may assume this fact without proving it.

(a) Prove that $P(X_n \geq cn \quad infinitely \quad often) = 1$ for all $c > 0$.

(b) Prove that for all $c > 0$

$$P(\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j \geq c) = 1$$

(c) Prove that

$$P(\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j = \infty) = 1$$

Solution:

(a) Since they are identically distributed, $P(X_1 \geq cn) = P(X_n \geq cn)$. So

$$\sum_{n=1}^{\infty} P(X_n \geq cn) = \infty$$

By the Borel Cantelli Lemma the statement in (a) holds.

(b) If ω is in the event $\{X_n \geq cn \quad infinitely \quad often\}$ then there is a subsequence n_k (which depends on ω) such that $X_{n_k} \geq cn_k$. So we have for all k

$$\frac{1}{n_n}\sum_{j=1}^{n_k} X_j \geq c$$

Hence the statement in (b) holds.

(c)

$$\{\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j = \infty\} = \bigcap_{N=1}^{\infty} \{\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} X_j \geq N\}$$

Each event in this countable intersection has probability one, so the intersection has probability one. (Countable union of measure zero sets is measure zero.)